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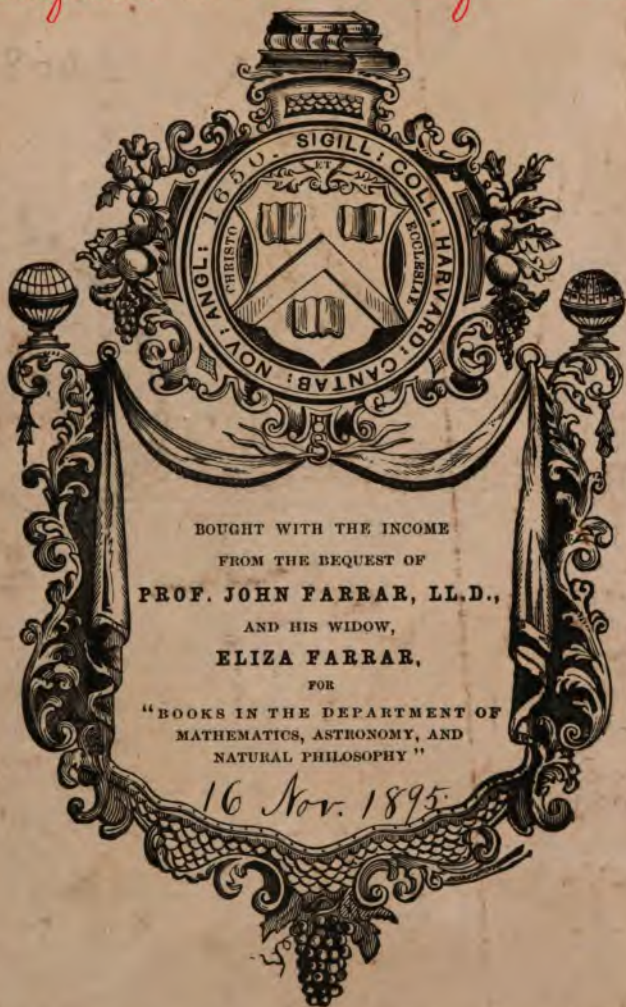
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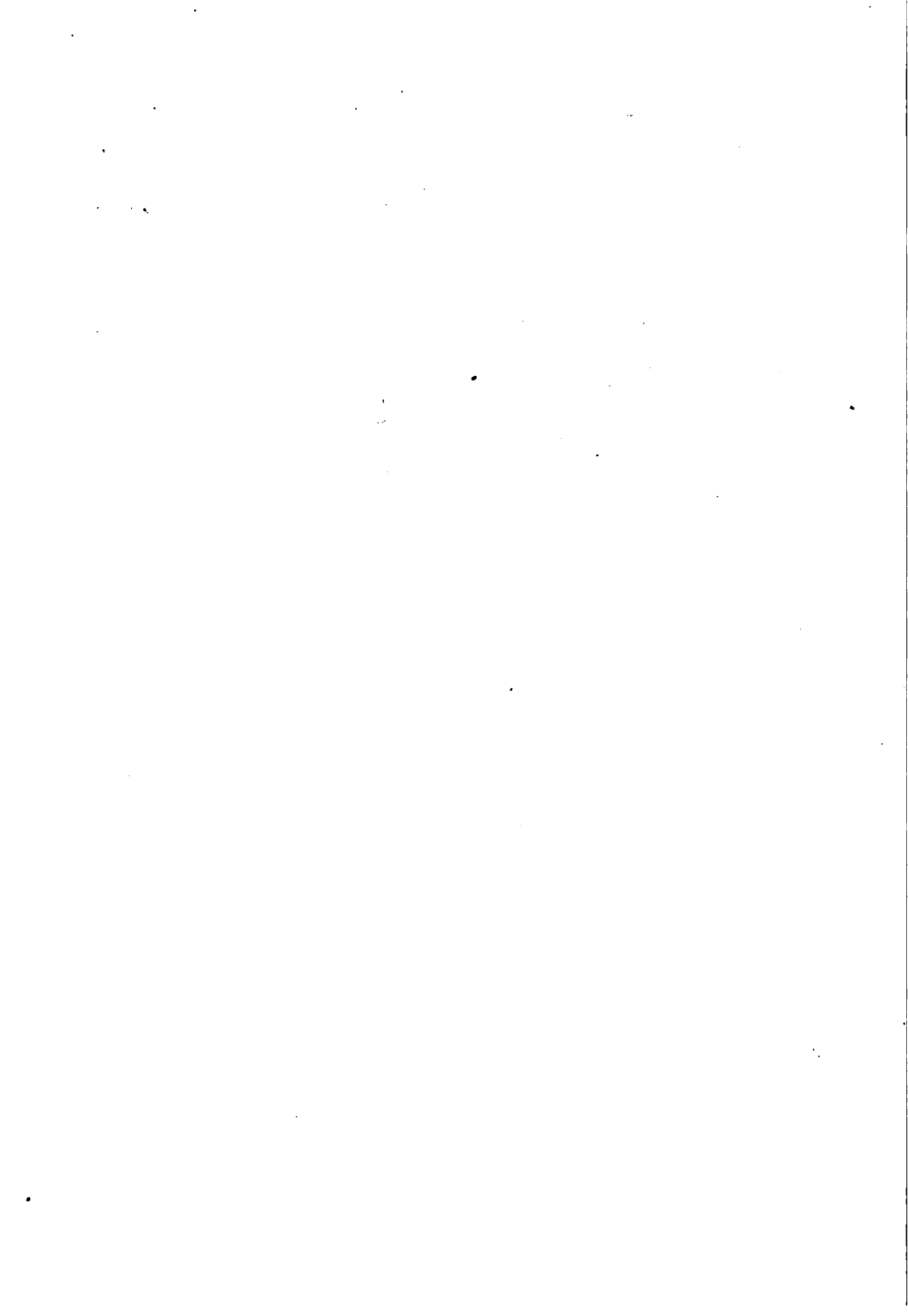


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AN  
ELEMENTARY TREATISE  
ON THE  
DIFFERENTIAL CALCULUS

FOUNDED ON THE  
METHOD OF RATES OR FLUXIONS

BY  
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## PREFACE.

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IN preparing this abridgment of their treatise on the Differential Calculus, the authors have endeavored to adapt it to the wants of those instructors who find the larger work too extensive for the time allotted to this subject.

J. M. R.  
W. W. J.

ANNAPOLIS, MARYLAND,  
*August, 1880.*

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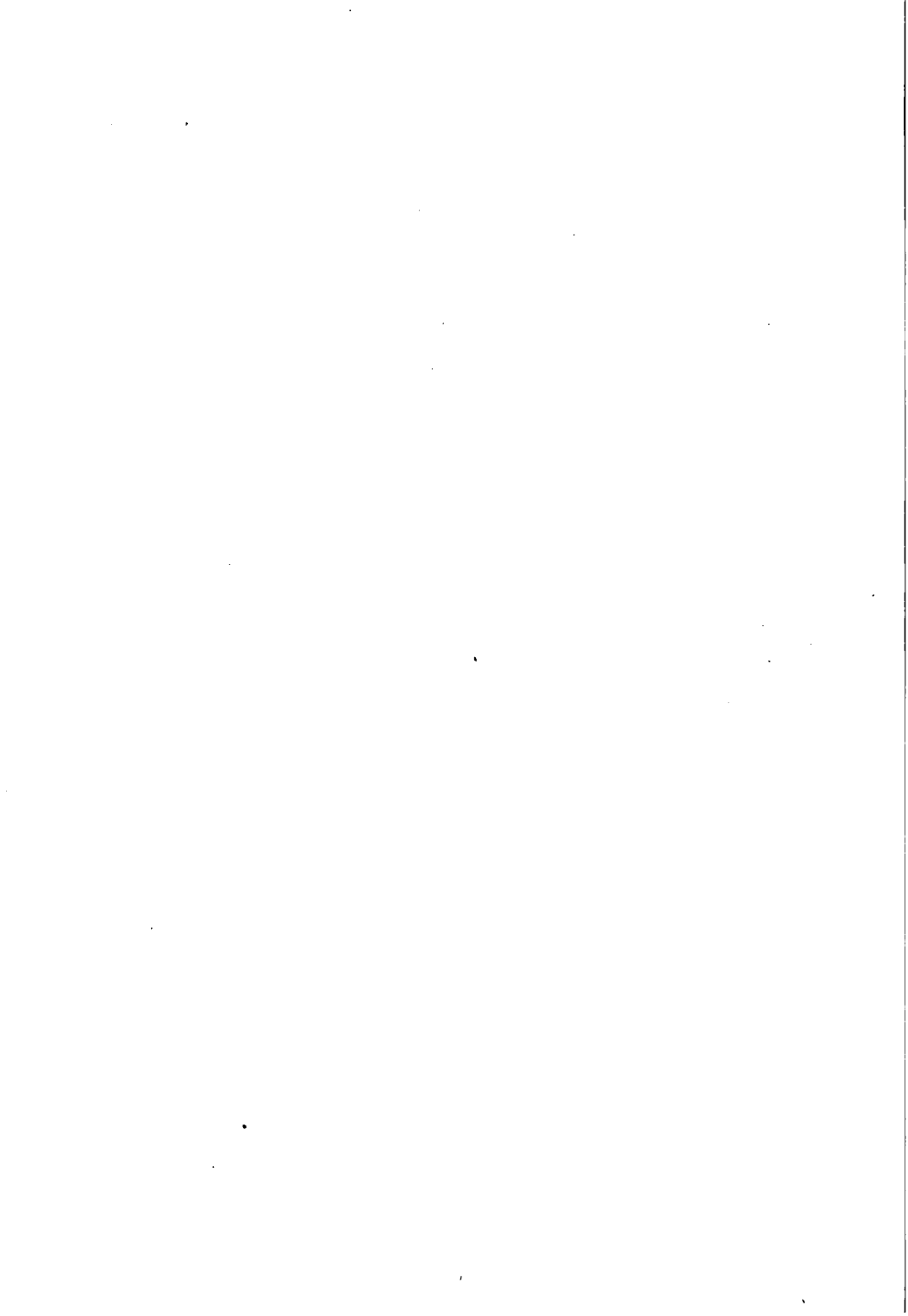
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# THE DIFFERENTIAL CALCULUS.

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## CHAPTER I.

### FUNCTIONS, RATES, AND DERIVATIVES.

---

#### I.

#### *Functions.*

1. A QUANTITY which depends for its value upon another quantity is said to be a *function* of the latter quantity. Thus  $x^2$ ,  $\tan x$ ,  $\log(a+x)$ , and  $a^x$  are functions of  $x$ .

The quantity upon which the function depends must be regarded as variable, and be represented in the analytical expression for the function by an algebraic symbol. This quantity is called the *independent variable*. It is essential that variation of the independent variable should actually produce variation of the function. Thus the quantities  $x^2$ ,  $x^2 + (a+x)(a-x)$ , and  $(\tan x + \cot x) \sin 2x$  are *not* functions of  $x$ , since each admits of expression in a form which does not involve  $x$ .

2. The notation  $f(x)$  is employed to denote any function of  $x$ , and, when several functions of  $x$  occur in the same in-

vestigation, such expressions as  $F(x)$ ,  $F'(x)$ ,  $\phi(x)$ , etc., are employed, the enclosed letter always denoting the independent variable. When expressions like  $f(1)$ ,  $f(a)$ ,  $f(2x)$ , or  $f(0)$  are employed, it must be understood that the enclosed quantity is to be substituted for  $x$  in the expression which defines  $f(x)$ . Thus, if we have

$$f(x) = x^2 + x,$$

$$f(1) = 2, \quad f(2x) = 4x^2 + 2x, \quad \text{and} \quad f(0) = 0.$$

$$\text{Again, if} \quad F(x) = \log_e x \quad (a > 1)$$

$$F(1) = 0, \quad F(0) = -\infty, \quad \text{and} \quad F(a) = 1.$$

**3.** When  $x$  denotes the independent variable upon which a function depends, any quantity independent of  $x$  is, in contradistinction, called a *constant*; both when it is an absolute constant, like 1,  $\sqrt{2}$ , or  $\pi$ , and when it is denoted by a symbol, like  $a$ ,  $u$ , or  $y$ , to which any value can be assigned. Thus, when  $a^x$  is denoted by  $f(x)$ , it is considered simply as a function of  $x$ , and  $a$  is regarded as a constant.

When it is desired to express that a quantity is a function of two quantities, both the symbols denoting them are placed between marks of parenthesis. Thus, since  $a^x$  is a function of  $x$  and  $a$ , we may write

$$f(x, a) = a^x.$$

Accordingly we have

$$f(y, b) = b^y, \quad f(3, 2) = 8, \quad \text{and} \quad f(2, 3) = 9.$$

**4.** It is often convenient to represent the value of a function of  $x$  by a single letter; thus, for example,  $y = x^2$ . When this notation is used, if we represent the independent variable  $x$  by the abscissa of a point, and the function  $y$  by the corre-

sponding ordinate, a curve may be constructed which will graphically represent the function, and will serve to illustrate its peculiarities.

Rectangular coordinates are usually employed for this purpose. See diagram, Art. 10.

A function of the form

$$y = mx + b,$$

$m$  and  $b$  being constants, is represented by a straight line. Functions of this form are, for this reason, called *linear functions*.

### *Implicit Functions.*

5. When an equation is given involving two variables  $x$  and  $y$ , either variable is obviously a function of the other; and the former variable, when its value is not *directly* expressed in terms of the other, is said to be an *implicit* function of the latter. Thus, if we have

$$ax^3 - 3axy + y^3 - a^3 = 0,$$

either variable is an implicit function of the other.

By solving the above equation for  $x$ , we obtain

$$x = \frac{3y}{2} \pm \sqrt{\left(a^3 + \frac{9y^3}{4} - \frac{y^3}{a}\right)}.$$

In this form of the equation,  $x$  is said to be an *explicit* function of  $y$ .

This example will serve to illustrate the fact, that from a single equation involving two variables, there may be derived two or more explicit functions of the same variable. In the above case,  $x$  is said to be a *two-valued* function of  $y$ ; while, since the equation is of the third degree in  $y$ , the latter is a *three-valued* function of  $x$ .

### *Inverse Functions.*

6. If  $y = f(x)$ ,  $x$  is some function of  $y$ ; we may therefore write

$$y = f(x), \quad \text{whence} \quad x = \phi(y).$$

Each of the functions  $f$  and  $\phi$  is then said to be the *inverse function* of the other. Thus, if

$$y = a^x, \quad \text{we have} \quad x = \log_a y;$$

hence each of these functions is the inverse of the other. So also the square and the square root are inverse functions.

7. In the case of the trigonometric functions, a peculiar notation for the inverse functions has been adopted. Thus, if we have

$$x = \sin \theta, \quad \text{we write} \quad \theta = \sin^{-1}x.$$

Whenever trigonometric functions are employed in the Calculus, the symbol representing the angle always denotes the *circular measure* of the angle; that is, the ratio of the arc to the radius. Hence  $\sin^{-1}x$  may be read either "the inverse sine of  $x$ ," or "the arc whose sine is  $x$ ."

The inverse trigonometric functions are evidently many-valued. See Art. 54.

### *The Classification of Functions.*

8. With reference to its *form*, an explicit function is either *algebraic* or *transcendental*.

An *algebraic function* is expressed by a definite combination of algebraic symbols, in which the exponents do not involve the independent variable.

All functions not algebraic are classed as *transcendental*. Under this head are included *exponential* functions; that is, those in which one or more exponents are functions of the variable, as, for example,  $a^x$ ,  $xa^{x^2}$ , etc.: logarithmic functions: the direct and inverse trigonometric functions, and other forms which arise in the higher branches of mathematics.

9. With reference to its mode of variation, a function is said to be an *increasing function* when it increases and decreases with  $x$ ; and a *decreasing function* when it decreases as  $x$  increases, and increases as  $x$  decreases. Thus, it is evident that  $x^3$  is always an increasing function of  $x$ , while  $\frac{1}{x}$  is always a decreasing function of  $x$ . Again,  $\tan x$  is always an increasing function, but  $\sin x$  is sometimes an increasing and sometimes a decreasing function of  $x$ .

10. The increase and decrease here considered are *algebraic*. For example,  $x^3$  is an increasing function when  $x$  is positive, but when  $x$  is negative it becomes a decreasing function; for, when  $x$  is negative and algebraically increasing,  $x^3$  is decreasing.

The curve  $y = x^3$  which illustrates this function is constructed in Fig. 1. Since algebraic increase in the value of  $x$  is represented by motion from left to right, whether the moving point is on the left or on the right of the axis of  $y$ , the downward slope of the curve on the left of the origin indicates that  $x^3$  is a decreasing function when  $x$  is negative.

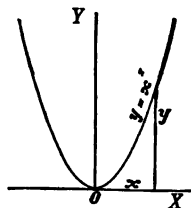


FIG. 1.

### *Expressions involving an Unknown Function.*

11. An expression involving  $f(x)$ , as, for example,  $xf(x)$  or  $F[f(x)]$ , is generally a function of  $x$ ; but it may happen

that such an expression has a value independent of  $x$ . Thus, suppose that, in the course of an investigation, the following equation presents itself:—

$$xf(x) = zf(z),$$

in which  $f$  denotes an unknown function, and  $x$  and  $z$  are entirely independent arbitrary quantities. When this is the case, we can make  $z$  a fixed quantity, and give to  $x$  any value whatever; that is, we can make  $x$  a variable and  $z$  a constant; but if  $z$  is a constant,  $zf(z)$  is likewise a constant, we can, therefore, write

$$xf(x) = c,$$

$c$  being an unknown constant. Hence we have

$$f(x) = \frac{c}{x}.$$

The value of the constant  $c$  is readily found, if we know the value of  $f(x)$  corresponding to any one value of  $x$ .

### Examples I.

1. ( $\alpha$ ) For what value of  $n$  does  $x^n$  cease to be a function of  $x$ ?
- ( $\beta$ ) For what values of  $x$  does it cease to be a function of  $n$ ?

( $\alpha$ ) When  $n = 0$ . ( $\beta$ ) When  $x = 1$ , or  $x = 0$ .

2. If  $y \left( 1 - \frac{a-x}{a+x} \right) = x + \frac{ax-x^2}{a+x}$ , show that  $y$  is a function of  $a$ , but not of  $x$ .

3. Show that  $\sin x \tan \frac{1}{2}x + \cos x$  is not a function of  $x$ .

4. If  $y = x + \sqrt{1+x^2}$ , show that  $y^2 - 2xy$  is not a function of  $x$ .

5. If  $f(x) = x^2$ , find the value of  $f(x+h)$ ; of  $f(2x)$ ; of  $f(x^2)$ ; of  $f(x^2-x)$ ; of  $f(1)$ ;  $f(12)$ ;  $f[f(x)]$ .

$$f(x+h) = x^2 + 2hx + h^2.$$

6. If  $f(\theta) = \cos \theta$ , find the value of  $f(0)$ ; of  $f(\frac{1}{2}\pi)$ ; of  $f(\frac{1}{4}\pi)$ ; of  $f(\pi)$ .

7. If  $F(x) = ax$ , give the value of  $F(a)$ ; of  $F(1)$ ; of  $F(0)$ . Also show that in this case  $[F(x)]^2 = F(2x)$ .

8. Given  $y^2 - 2ay + x^2 = 0$ , make  $y$  an explicit function of  $x$ .

$$y = a \pm \sqrt{a^2 - x^2}.$$

9. Given  $1 + \log_a y = 2 \log_a (x + a)$ , make  $y$  an explicit function of  $x$ .

$$y = \frac{(x+a)^2}{a}.$$

10. Given the equations—

$$n + 1 = n(\cos^2 \theta' + \cos \theta' \cos \theta + \cos^2 \theta),$$

and

$$n - 1 = n(\sin^2 \theta' + \sin \theta' \sin \theta + \sin^2 \theta);$$

eliminate  $n$ , and make  $\theta'$  an explicit function of  $\theta$ . Also make  $n$  an explicit function of  $\theta$ .

$$\theta' = \theta \pm \frac{1}{2}\pi, \text{ and } n = \mp \frac{1}{\sin \theta \cos \theta}.$$

11. Given  $\sin^{-1} x + \sin^{-1} y = \alpha$ , make  $y$  an explicit function of  $x$ .

$$y = \sin \alpha \sqrt{1 - x^2} - x \cos \alpha.$$

12. Given  $\tan^{-1} x + \tan^{-1} y = \alpha$ , make  $y$  an explicit function of  $x$ .

$$y = \frac{\tan \alpha - x}{1 + x \tan \alpha}.$$

13. Given  $xy - 2x + y = n$ , show that  $y$  is not a function of  $x$  when  $n = 2$ .

14. If  $y = \frac{2x-1}{3x-2}$ , show that the inverse function is of the same form.

15. If  $y = f(x) = \frac{1+x}{1-x}$ , find  $s = f(y)$ , and express  $s$  as a function of  $x$ .

$$s = -\frac{1}{x}.$$

16. If both  $f$  and  $\phi$  denote increasing functions, or, if both denote decreasing functions, show that  $\phi[f(x)]$  is an increasing function. Also show that the inverse of an increasing function is an increasing function.

17. Find the inverse of the function,  $y = \log_e [x + \sqrt{1+x^2}]$ .

$$x = \frac{1}{2} (e^y - e^{-y}).$$

18. If  $f(x)$  be an unknown function having the property

$$f(x) + f(y) = f(xy),$$

prove that

$$f(1) = 0.$$

Put  $y = 1$ .

19. If  $f(x)$  has the property

$$f(x+y) = f(x) + f(y),$$

prove that  $f(0) = 0$ . Also prove that the function has the property

$$f(\phi x) = \phi f(x),$$

in which  $\phi$  is a positive or negative integer.

For positive integers, put  $y = x, 2x, 3x$ , etc., in the given equation; for negative integers, put  $y = -x$ .

20. If  $f$  denotes the same function as in Example 19, prove that

$$f(mx) = mf(x),$$

$m$  denoting any fraction.

*Solution* :—

Putting  $z = \frac{p}{q}x$ ,

$$qz = \phi x,$$

$$f(qz) = f(\phi x);$$

hence, by Example 19,

$$qf(z) = \phi f(x),$$

or

$$f(z) = \frac{\phi}{q} f(x),$$

$\therefore$

$$f\left(\frac{p}{q}x\right) = \frac{p}{q} f(x).$$

21. Given, the property of the same function proved in Example 20; viz.,

$$f(mx) = mf(x);$$



by putting  $x$  for  $mx$ , show that

$$\frac{1}{x}f(x) = \frac{1}{x}f(x),$$

and thence deduce the form of the function. See Art. 11.

22. Given,  $[\phi(x)]^x = [\phi(x)]^x$ , and  $\phi(1) = e$ ,  $f(x) = cx$ .  
determine  $\phi(x)$ .  $\phi(x) = e^{\frac{1}{x}}$

23. Given  $\phi(x) + \phi(y) = \phi(xy)$   
prove  $\phi(x^m) = m\phi(x)$ ,  
and thence prove  $\phi(x) = c \log x$ .

Use the methods of Examples 19, 20, and 21.

## II.

### Rates.

12. In the Differential Calculus, variable quantities are regarded as undergoing continuous variation in magnitude, and the *rates* of variation, denoted by appropriate symbols, are employed in connection with the values of the variables themselves.

If a varying quantity be represented by the distance of a point moving in a straight line from a fixed origin taken on that line, the velocity of the moving point will represent the rate of increase or decrease of the varying quantity.

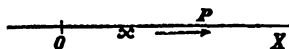


FIG. 2.

Thus  $O$  (Fig. 2) being the fixed origin and  $OP$  a variable denoted by  $x$ ,  $P$  is the moving point whose velocity represents the rate of  $x$ . The velocity of  $P$ , or the rate of  $x$ , is regarded as positive when  $P$  moves in the direction in which  $x$  increases algebraically; thus, taking the direction  $OX$ , or toward the right, as the positive direction in laying off  $x$ , the

velocity is positive when  $P$  moves toward the right, whether its position be on the right or on the left of the origin. Accordingly, a rate of *algebraic decrease* is considered as negative, and would be represented by a point moving toward the left.

### *Constant Rates.*

13. The rate of a quantity like the velocity of a point may be either constant or variable. A velocity is uniform or constant, when the spaces passed over in any equal intervals of time are equal, or, in other words, *when the spaces passed over in any intervals of time are proportional to the intervals.*

The numerical measure of a uniform velocity is *the space passed over in a unit of time*; then if  $t$  denote the time elapsed from an assumed origin of time, and  $k$  the space passed over by a moving point in a unit of time,  $kt$  will denote the space passed over in the time  $t$ . Hence, whenever the velocity is uniform, the quotient obtained by dividing the number of units of space by the number of units of time occupied in describing this space is constant, and serves as the numerical measure of the velocity.

14. Now, if  $x$  be a quantity having a uniform rate  $k$ , it will be represented by the distance from the origin of a point having the uniform velocity  $k$ , and if  $a$  denote the value of  $x$  when  $t$  is zero, we shall have

$$x = a + kt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This formula expresses a uniformly varying quantity as a function of  $t$ . When  $x$  is a uniformly decreasing quantity,  $k$  is, of course, negative.

Conversely, if  $x$ , when expressed as a function of  $t$ , is of the form (1), involving the first power only of  $t$ , then  $x$  is a quantity having a uniform rate, and the coefficient  $k$  is a measure of this rate.

*Variable Velocities.*

15. If the velocity of a point be *not* uniform, *its numerical measure at any instant is the number of units of space which would be described in a unit of time, were the velocity to remain constant from and after the given instant.*

Thus, when we speak of a body as having at a given instant a velocity of 32 feet per second, we mean that should the body continue to move during the whole of the next second, with the same velocity which it had at the given instant, 32 feet would be described. The *actual* space described may be greater or less, in consequence of the change in velocity which takes place during the second ; it is, for instance, greater than the measure of the velocity at the beginning of the second, in the case of a falling body, because the velocity increases throughout the second.

16. Attwood's machine for determining experimentally the velocities acquired by falling bodies furnishes a familiar example of the practical application of the principle embodied in the above definition.

This apparatus consists essentially of a thread passing over a fixed pulley, and sustaining equal weights at each extremity, the pulley being so constructed as to offer but slight resistance to turning. On one of the weights a small bar of metal is placed, which, destroying the equilibrium, causes the weight to descend with an increasing velocity. To determine the value of this velocity at any point, a ring is so placed as to intercept the bar at that point, and allow the weight to pass. Thus, the sole cause of the variation of the velocity having been removed, the weight moves on uniformly with the required velocity, and the space described during the next second becomes the measure of this velocity.

*Variable Rates.*

17. When  $x$  is a function of  $t$ , but not of the form expressed by equation (1), Art. 14—that is, when the function is not linear—the rate of  $x$  will be variable. To obtain the measure of this rate at any given instant, we employ the same principle as in the case of a variable velocity. Thus, let  $x$  be represented by  $OP$  as in Fig. 2, Art. 12, let the symbol  $dt$  denote an assumed interval of time, and let  $dx$  denote the space which would be described in the time  $dt$ , were  $P$  to move with the velocity which it has at the given instant unchanged throughout the interval of time  $dt$ . Then the space which would be described in a unit of time is, evidently,

$$\frac{dx}{dt},$$

which is therefore the measure of the velocity of  $P$ , or the rate of  $x$ .

This ratio is in general variable, but, when  $x$  is of the form  $a + kt$ , it has been shown in Art. 14 that  $k$  is the measure of the rate; we therefore have

$$\frac{dx}{dt} = k, \quad \text{when} \quad x = a + kt.$$

*Differentials.*

18. The quantities  $dx$  and  $dt$  are called respectively “the differential of  $x$ ” and “the differential of  $t$ .”

In accordance with the definition of  $dx$  given in the preceding article, the *differential* of a variable quantity at any instant is the increment which would be received in the time  $dt$ , were the quantity to continue to increase uniformly during that interval of time with the rate it has at the given

instant. *The quotient obtained by dividing the differential of any quantity by  $dt$  is therefore the measure of the rate of the quantity.*

The differential of a quantity is denoted by prefixing  $d$  to the symbol denoting the quantity; when the symbol denoting the quantity is not a single letter it is usually enclosed by marks of parenthesis to avoid ambiguity. Thus,  $d(x^2)$ ,  $d(xy)$ ,  $d(\tan x)$ ,  $d(ax + x^2)$ , etc.

### *The Differentials of Polynomials.*

19. Let  $x$  and  $y$  denote two variable quantities, and let  $a$  and  $b$  denote particular simultaneous values of  $x$  and  $y$ , while  $k$  and  $k'$  denote corresponding values of the rates of  $x$  and  $y$ .

Now, if  $x$  and  $y$  should continue to vary with these rates, their values would (see Art. 14) be expressed by

$$x = a + kt,$$

and

$$y = b + k't,$$

whence

$$x + y = a + b + (k + k')t.$$

Thus the quantity  $x + y$  would become a uniformly varying quantity, and, by Art. 14, its rate would be  $k + k'$ , which, therefore, is the measure of the rate of  $x + y$  at the instant when  $x$  and  $y$  have the rates  $k$  and  $k'$ . Consequently,

$$\frac{d(x+y)}{dt} = k + k' = \frac{dx}{dt} + \frac{dy}{dt}.$$

Now, since  $k$  and  $k'$  denote any values of the rates, this equation is universally true. We have, therefore,

$$d(x+y) = dx + dy. \quad \dots \dots (1)$$

This formula is easily extended to the sum of any number of variables. Thus,

$$d(x+y+z+\dots) = dx + d(y+z+\dots) = dx + dy + dz + \dots \dots (2)$$

20. The differential of a constant is evidently zero, hence

$$d(x+h) = dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, if  $y = -x$ ,  $y + x = 0$ ,

hence, by equation (1), since zero is a constant, we have

$$dy + dx = 0, \quad \text{or} \quad dy = -dx;$$

that is,  $d(-x) = -dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

The differential of a negative term is therefore the negative of the differential of the term taken positively.

It appears, on combining the results expressed in equations (2), (3), and (4), that *the differential of a polynomial is the algebraic sum of the differentials of its terms; and that constant terms disappear from the result.*

### *The Differential of a Term having a Constant Coefficient.*

21. Let the term be denoted by  $mx$ ,  $m$  denoting a constant.

Resuming equation (2), Art. 19; viz.,

$$d(x+y+z+\dots) = dx+dy+dz+\dots,$$

and denoting the number of terms by  $p$ , we put

$$x=y=z=\dots,$$

thus obtaining  $d(px) = p dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

$p$  denoting an integer.

To extend equation (1) to the case in which  $m$  denotes a fraction, let

$$z = \frac{p}{q} x, \quad \text{then} \quad qz = px.$$

By applying equation (1) we obtain

$$q dz = p dx, \quad \text{or} \quad dz = \frac{p}{q} dx;$$

that is, 
$$d\left(\frac{p}{q} x\right) = \frac{p}{q} dx.$$

Hence generally, when  $m$  is positive,

$$d(mx) = m dx. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since  $d(-x) = -dx$ , this equation is true likewise when  $m$  is negative.

It therefore follows that *the differential of a term having a constant coefficient is equal to the product of the differential of the variable factor by the constant coefficient.*

### Examples II.

1. Find the differential of  $\frac{2x}{3a}$ , and of  $\frac{x}{m-2}$ .  $\frac{2dx}{3a}$ , and  $\frac{dx}{m-2}$ .
2. Find the differential of  $\frac{x-a}{m^2}$ , and of  $\frac{a-x}{m^2}$ .  $\frac{dx}{m^2}$ , and  $-\frac{dx}{m^2}$ .
3. Find the differential of  $\frac{a+b+(a-b)x}{a^2-b^2}$ .  $\frac{dx}{a+b}$ .
4. Find the differential of  $\frac{a+x}{a+b}$ , and of  $\frac{b(x+y)}{a(a+b)}$ .  $\frac{dx}{a+b}$ , and  $\frac{b(dx+dy)}{a(a+b)}$ .

5. Given  $ay + bx + 2cx + ab = 0$ , to find  $\frac{dy}{dx}$ .  $\frac{dy}{dx} = -\frac{b+2c}{a}$ .

6. Given  $y \log a + x \sin \alpha - y \cos \alpha - ax + \tan \alpha = 0$ , to find  $\frac{dy}{dx}$ .  
 $\frac{dy}{dx} = \frac{a - \sin \alpha}{\log a - \cos \alpha}$ .

7. Given  $ay \cos^2 \alpha - 2b(1 - \sin \alpha)x = b(a - x \cos^2 \alpha)$ , to find  $\frac{dy}{dx}$ .  
 $\frac{dy}{dx} = \frac{b(1 - \sin \alpha)}{a(1 + \sin \alpha)}$ .

8. Given  $a^2 + 2(1 + \cos \alpha)y = (x + y) \sin^2 \alpha$ , to find  $\frac{dy}{dx}$ .  
 $\frac{dy}{dx} = \tan^2 \frac{\alpha}{2}$ .

9. Given  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ , to express  $dz$  in terms of  $dx$  and  $dy$ .

$$dz = -\frac{c}{a} dx - \frac{c}{b} dy.$$

10. A man whose height is 6 feet walks directly away from a lamp-post at the rate of 3 miles an hour. At what rate is the extremity of his shadow travelling, supposing the light to be 10 feet above the level pavement on which he is walking?

*Draw a figure, and denote the variable distance of the man from the lamp-post by  $x$ , and the distance of the extremity of his shadow from the post by  $y$ .*

7½ miles per hour.

11. At what rate does the man's shadow (Ex. 10) increase in length?

### III.

#### *Differentials of Functions of an Independent Variable.*

22. When the variables involved in any mathematical investigation are functions of an independent variable  $x$ , the latter may be assumed to have a rate denoted by  $\frac{dx}{dt}$ , in which



$dx$  is arbitrary. So also the corresponding rate of  $y$  will be denoted by  $\frac{dy}{dt}$ , and, if  $y$  is a function of  $x$ , the value of  $dy$  will depend in part upon the assumed value of  $dx$ .

To *differentiate* a function of  $x$  is to express its differential in terms of  $x$  and  $dx$ .

It is to be understood, of course, that the differentials involved in an equation are all taken with reference to the same value of  $dt$ .

If two quantities are always equal, their simultaneous rates are evidently equal; and hence their differentials are likewise equal. We can therefore *differentiate an equation*; that is, express the equality of the differentials of its members; provided the equation is true for all values of the variables involved. Thus, from the identical equation

$$(x+h)^2 = x^2 + 2hx + h^2,$$

it follows that  $d[(x+h)^2] = d(x^2) + 2h dx$ .

### *The Derivative.*

**23.** Before proceeding to the differentiation of the various functions of  $x$ , it is necessary to show that, if

$$y = f(x), \quad \dots \dots \dots (1)$$

the ratio  $\frac{dy}{dx}$

has a definite value for each value of  $x$ , independent of the assumed value of  $dx$ .

Let a particular value of  $x$  be denoted by  $a$ , and let the corresponding value of  $dx$  be an arbitrary quantity.

Now, although  $dx$  is arbitrary, since  $dt$  is likewise arbitrary, the rate of  $x$ , that is, the ratio

$$\frac{dx}{dt}, \quad \dots \dots \dots (2)$$

may be assumed to have a certain *fixed value* at the instant when  $x = a$ . The corresponding value of the rate of  $y$ , denoted by

$$\frac{dy}{dt}, \quad . . . . . (3)$$

evidently depends solely upon the rate of  $x$  and upon the form of the function  $f$  in equation (1). Hence, when the value of the rate (2) is fixed, the value of (3) is also definitely fixed.

Denoting these fixed values by  $k$  and  $k'$ , we have, when  $x = a$ ,

$$\frac{dx}{dt} = k, \text{ and } \frac{dy}{dt} = k', \text{ whence } \frac{dy}{dx} = \frac{k'}{k}.$$

Hence, corresponding to a particular value  $a$  of  $x$ , there exists a determinate value  $\frac{k'}{k}$ , of the ratio  $\frac{dy}{dx}$ , notwithstanding the fact that  $dx$  has an arbitrary value; in other words, *the value of the ratio  $\frac{dy}{dx}$  is independent of the arbitrary value of  $dx$ .*

**24.** It is obvious that, in general, this ratio will have different values corresponding to different values of  $x$ , and hence that it may be expressed as a function of  $x$ , and denoted by  $f'(x)$ ; thus,—

$$\frac{dy}{dx} = f'(x). \quad . . . . . (1)$$

The form of this new function  $f'$  will evidently depend upon that of the given function  $f$ .

The function  $f'(x)$  is called the *derivative* of  $f(x)$ , and, since equation (1) may be written in the form

$$dy = f'(x) dx,$$

it is also called the *differential coefficient* of  $y$  regarded as a function of  $x$ .

When, however, the given function  $f(x)$  is of the linear form

$$y = mx + b,$$

the derivative is no longer a function of  $x$ , but is a constant, since the value of  $y$  gives

$$dy = m dx,$$

or 
$$\frac{dy}{dx} = m.$$

### *The Geometrical Meaning of the Derivative.*

**25.** Representing the corresponding values of  $x$  and  $y$  by the rectangular coordinates of a moving point, if this point move in a uniform direction, so as to describe a straight line,—that is, if  $y$  be a linear function of  $x$ ,—the value of  $\frac{dy}{dx}$  will be constant, by the preceding article. Hence, in the general case, when this ratio is variable, the point will move in a variable direction.

If we denote the inclination of this direction to the axis of  $x$  by  $\phi$ , the value of  $\phi$  will vary with the value of  $x$ , and the point will describe a curve.

The tangent line to a curve is defined as follows:—

*The tangent to a curve at any point is the straight line which passes through the point, and has the direction of the curve at that point.\**

Hence, for any point of the curve,  $\phi$  denotes the inclination to the axis of  $x$  of the tangent line at that point.

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\* It will be shown hereafter (Art. 49) that, in the case of the circle, this general definition of a tangent line agrees with that usually given in Plane Geometry.

**26.** Now, if a point, at first moving in the curve, should, after passing the point whose abscissa is  $a$ , so move that the rates  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  retain the values which they had at the instant of passing the given point, the direction of its motion will become constant, and the point will describe a straight line tangent to the curve at the given point.

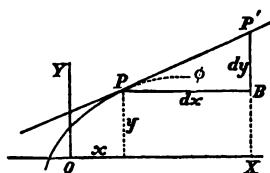


FIG. 3.

The value of  $dx$  may be represented by an arbitrary increment of  $x$  as in Fig. 3; the value of  $dy$  will then be represented by the corresponding increment which would be received by  $y$ , were the point moving in the tangent line, as indicated in

the diagram. Hence

$$\frac{dy}{dx} = \tan \phi,$$

which is evidently independent of the assumed value of  $dx$ .\* It follows that the value of the derivative of  $f(x)$ , for any value of  $x$ , is represented by the trigonometric tangent of the inclination to the axis of  $x$  of the curve  $y = f(x)$ , at the point corresponding to the given value of  $x$ .

**27.** The moving point, which is conceived to describe the curve, may pass over it in either of two directions differing by  $180^\circ$ . The two corresponding values of  $\phi$  give, however, the same value of  $\tan \phi$ , since  $\tan(\phi \pm 180^\circ) = \tan \phi$ .

Thus, in Fig. 3, the point  $P$  may be regarded as moving so as to increase  $x$  and  $y$ , in which case both  $dx$  and  $dy$  will be positive, and  $\phi$  will be in the first quadrant; or  $P$  may

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\* In other words, the value of the derivative is determined by the form of the function  $f$  which determines the curve, and the value of  $x$  which fixes the position of  $P$ .

move in the opposite direction, making  $dx$  and  $dy$  negative, and placing  $\phi$  in the third quadrant. In either case,  $\frac{dy}{dx}$  or  $\tan \phi$  is positive.

**28.** It is evident that when  $f(x)$  is an increasing function, as in Fig. 3,  $\frac{dy}{dx}$  is positive, and that when it is a decreasing function,  $\frac{dy}{dx}$  is negative.

Thus the sign of  $f'(x)$  for any value of  $x$  is positive or negative according as  $f(x)$  is, for that value of  $x$ , an increasing or a decreasing function. For example, it is evident that the value of the derivative of  $\sin x$  must be positive when  $x$  is between 0 and  $\frac{1}{2}\pi$ , negative when  $x$  is between  $\frac{1}{2}\pi$  and  $\frac{3}{2}\pi$ , and so on.

When the notation  $\frac{dy}{dx}$  is used, the value of the derivative corresponding to a particular value  $a$  of  $x$  is expressed by  $\left[\frac{dy}{dx}\right]_a$  which is equivalent to  $f'(a)$ . See Art. 2.

### Examples III.

1. If a point move in the straight line  $2y - 7x - 5 = 0$ , so that its ordinate decreases at the rate of 3 units per second, at what rate is the point moving in the direction of the axis of  $x$ ?

$$\frac{dx}{dt} = -\frac{6}{7}.$$

2. If a point starting from  $(0, b)$  move so that the rates of its co-ordinates are  $k$  and  $k'$ , show that its path is  $y = mx + b$ ,  $m$  being equal to  $\frac{k'}{k}$ .

*Express  $x$  and  $y$  in terms of  $t$  (Art. 14), and eliminate  $t$ .*

3. If a point moving in a curve passes through the point  $(5, 3)$

moving at equal rates upward and toward the left, find the value of  $\left[\frac{dy}{dx}\right]_1$ , also the equation of the tangent line to the curve at the given point.  $\left[\frac{dy}{dx}\right]_1 = -1$ , and  $y+x=8$ .

4. If a point is moving in the straight line

$$x \cos \alpha + y \sin \alpha = p,$$

its rate in the positive direction of the axis of  $x$  being  $l \sin \alpha$ , what is its rate of motion in the direction of the axis of  $y$ ?

$$-l \cos \alpha.$$

5. Given  $a y \sin \alpha - a x + a x \cos \alpha - b^2 \sec \alpha = 0$ ; show that  $\phi$  is constant and equal to  $\frac{1}{2}\alpha$ .

6. If  $f(x) = \tan x$ , show that  $f''(x)$  must always be positive.

7. Show, by tracing the curve, that if  $y = x^3$ ,  $\frac{dy}{dx}$  can never be negative.

## CHAPTER II.

### THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

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#### IV.

#### *The Square.*

**29.** IN establishing the formulas for the differentiation of the simple algebraic functions of an independent variable, we find it convenient to begin with the square. The object of this article is, therefore, to express  $d(x^2)$  in terms of  $x$  and  $dx$ .

We first deduce a relation between two values of the derivative of the function and the corresponding values of the independent variable; for this purpose, we assume two values of the variable having a constant ratio  $m$ . Thus, if

$$z = m x,$$

$$z^2 = m^2 x^2.$$

Differentiating by equation (2), Art. 21,

$$dz = m dx, \quad \text{and} \quad d(z^2) = m^2 d(x^2);$$

dividing, we obtain

$$\frac{d(z^2)}{dz} = m \frac{d(x^2)}{dx}.$$

Whence, dividing by  $z = m x$  to eliminate  $m$ , we have

$$\frac{1}{z} \cdot \frac{d(z^2)}{dz} = \frac{1}{x} \cdot \frac{d(x^2)}{dx}. \quad \dots \dots (1)$$

The derivatives  $\frac{d(z^2)}{dz}$  and  $\frac{d(x^2)}{dx}$  are, by Art. 23, functions of  $z$  and of  $x$  respectively, independent of the values of  $dz$  and  $dx$ ; moreover, equation (1) is true for all values of  $x$  and  $z$ , these quantities being entirely independent of each other, since the arbitrary ratio  $m$  has been eliminated. Therefore, either of these quantities may be assumed to have a fixed value, while the other is variable; hence it follows that the value of each member of this equation must be a fixed quantity, independent of the value of  $x$  or of  $z$ . Denoting this fixed value by  $c$ , we therefore write

$$\frac{1}{x} \cdot \frac{d(x^2)}{dx} = c,$$

or

$$d(x^2) = cx dx \quad . \quad . \quad . \quad . \quad . \quad (2)$$

**30.** To determine the unknown constant  $c$ , we apply this result to the identity

$$(x + h)^2 = x^2 + 2hx + h^2.$$

Differentiating each member (Art. 22) by equation (2), we have

$$c(x + h) d(x + h) = cx dx + 2h dx;$$

since  $d(x + h) = dx$ , this equation reduces to

$$ch dx = 2h dx,$$

or

$$(c - 2)h dx = 0.$$

Now, since  $h$  and  $dx$  are arbitrary quantities, this equation gives

$$c = 2;$$

this value of  $c$  substituted in (2) gives

$$d(x^2) = 2x dx. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

That is, *the differential of the square of a variable equals twice the product of the variable and its differential.*



31. Employing the derivative notation, this result may also be expressed thus :—

$$\text{If} \quad f(x) = x^2, \quad f'(x) = 2x.$$

This derivative is negative for negative values of  $x$ , therefore, for these values,  $x^2$  is a decreasing function, as already mentioned (Art. 10) in connection with the curve illustrating this function.

Since  $x$  and  $dx$  are arbitrary, we may substitute for them any variable and its differential. Equation (a) therefore enables us to differentiate the square of any variable whose differential is known. Thus,—

$$d(5x - 3)^2 = 2(5x - 3)5dx = 10(5x - 3)dx.$$

$$\begin{aligned} \text{Again,} \quad d(ax^2 + bx)^2 &= 2(ax^2 + bx) d(ax^2 + bx) \\ &= 2(ax^2 + bx)(2ax + b) dx. \end{aligned}$$

### *The Square Root.*

32. To derive the differential of the *square root*, we put

$$\begin{aligned} y &= \sqrt{x}, \\ \text{whence} \quad y^2 &= x; \end{aligned}$$

$$\text{differentiating by (a),} \quad 2y dy = dx,$$

$$\text{or} \quad dy = \frac{dx}{2y}.$$

$$\text{But } y = \sqrt{x}, \therefore d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}. \quad \dots \dots \dots (b)$$

That is, *the differential of the square root of a variable is equal to the quotient arising from dividing the differential of the variable by twice the given square root.*

Thus, 
$$d[\sqrt[4]{(a^3 - x^3)}] = \frac{-x dx}{\sqrt[4]{(a^3 - x^3)}},$$

or, using derivatives,

$$\frac{d[\sqrt[4]{(a^3 - x^3)}]}{dx} = -\frac{x}{\sqrt[4]{(a^3 - x^3)}}.$$

### Examples IV.

1. Differentiate  $(2x + 3)^3$ , and find the numerical value of its rate, when  $x$  has the value 8, and is decreasing at the rate of 2 units per second.

*The differential required is denoted by  $d[(2x + 3)^3]$ , and the rate by  $\frac{d[(2x + 3)^3]}{dt}$ ; the given rate  $\frac{dx}{dt} = -2$ .*

152 units per second.

2. Find the numerical value of the rate of  $(x^3 - 2x)^3$ , when  $x = 3$ , and is increasing at the rate of  $\frac{1}{4}$  of one unit per second.

*Differentiate the given expression before substituting.*

12 units per second.

3. Find the numerical value of the rate of  $\sqrt[4]{(y^3 + x^3)}$ , when  $y = 7$  and  $x = -7$ , if  $y$  is increasing at the rate of 12 units per second, and  $x$  at the rate of 4 units per second.

4  $\sqrt[4]{2}$  units per second.

4. If  $f(x) = x - \sqrt[4]{(x^3 - a^3)}$ , find  $f'(x)$ , and show that  $f(x)$  is a decreasing function.

$$f'(x) = 1 - \frac{x}{\sqrt[4]{(x^3 - a^3)}}.$$

5. Differentiate the identity  $(\sqrt[4]{x} + \sqrt[4]{a})^4 = x + a + 2\sqrt[4]{ax}$ , and show that the result is an identity.

6. Differentiate  $\sqrt[4]{\left(\frac{x^3 - 2ax}{a^3 - 2ab}\right)}$ .

*The constant factor  $\frac{1}{\sqrt[4]{(a^3 - 2ab)}}$  should be separated from the variable factor before differentiation.*

$$\frac{1}{\sqrt[4]{(a^3 - 2ab)}} \cdot \frac{x - a}{\sqrt[4]{(x^3 - 2ax)}} dx.$$

$$7. \text{ If } f(x) = (1 + x^2)^{\frac{1}{2}}, \quad f'(x) = \frac{x}{(1 + x^2)^{\frac{1}{2}}}.$$

$$8. \text{ If } f(x) = \sqrt[4]{(a^2 + 2b^2x + cx^2)}, \quad f'(x) = \frac{b^2 + cx}{\sqrt[4]{(a^2 + 2b^2x + cx^2)}}.$$

$$9. \text{ If } f(x) = \sqrt[4]{x + \sqrt{1 + x^2}}, \quad f'(x) = \frac{\sqrt[4]{x + \sqrt{1 + x^2}}}{2\sqrt[4]{1 + x^2}}.$$

$$10. \text{ If } f(x) = \frac{a^3}{x - \sqrt{x^2 - a^2}}, \quad f'(x) = 1 + \frac{x}{\sqrt{x^2 - a^2}}.$$

*Rationalize the denominator before differentiating.*

$$11. \text{ Given } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ express } \frac{dy}{dx} \text{ in terms of } x, \text{ and give the values of } \left. \frac{dy}{dx} \right|_0 \text{ and } \left. \frac{dy}{dx} \right|_a. \quad \frac{dy}{dx} = \mp \frac{b}{a} \cdot \frac{x}{\sqrt{a^2 - x^2}}.$$

$$12. \text{ Given } y^2 = 4ax, \text{ express } \frac{dy}{dx} \text{ in terms of } x, \text{ also in terms of } y, \text{ and give the values of } \left. \frac{dy}{dx} \right|_a \text{ and } \left. \frac{dy}{dx} \right|_{4a}. \quad \frac{dy}{dx} = \sqrt{\frac{a}{x}} = \frac{2a}{y}.$$

13. A man is walking on a straight path at the rate of 5 ft. per second; how fast is he approaching a point 120 ft. from the path in a perpendicular, when he is 50 ft. from the foot of the perpendicular?

*Solution:—*

Let  $x$  denote the variable distance of the man from the foot of the perpendicular, so that  $\frac{dx}{dt}$  may denote the known velocity of the man, and let  $a$  denote the length of the perpendicular (120 ft.); then the distance of the man from the point is  $\sqrt{(a^2 + x^2)}$ , of which the rate of change is denoted by

$$\frac{d[\sqrt{(a^2 + x^2)}]}{dt} = \frac{x}{\sqrt{(a^2 + x^2)}} \frac{dx}{dt}.$$

At the instant considered,  $x = 50$  ft., while  $a = 120$  ft., and  $\frac{dx}{dt} = -5$  ft.

per second. By substituting these values, we obtain  $-1\frac{1}{8}$ . Hence his distance from the point is diminishing (that is, he is approaching it) at the rate of  $1\frac{1}{8}$  ft. per second.

14. If the side of an equilateral triangle increase uniformly at the rate of 3 ft. per second, at what rate per second is the area increasing, when the side is 10 ft. ? 15  $\sqrt{3}$  sq. ft.

15. A stone dropped into still water produces a series of continually enlarging concentric circles; it is required to find the rate per second at which the area of one of them is enlarging, when its diameter is 12 inches, supposing the wave to be then receding from the centre at the rate of 3 inches per second. 36  $\pi$  sq. inches.

16. If a circular disk of metal expand by heat so that the area  $A$  of each of its faces increases at the rate of 0.01 sq. ft. per second, at what rate per second is its diameter increasing ?  $\frac{1}{100\sqrt{\pi A}}$  ft.

17. A man standing on the edge of a wharf is hauling in a rope attached to a boat at the rate of 4 ft. per second. The man's hands being 9 ft. above the point of attachment of the rope, how fast is the boat approaching the wharf when she is at a distance of 12 ft. from it ? 5 ft. per second.

18. A ladder 25 ft. long reclines against a wall; a man begins to pull the lower extremity, which is 7 ft. distant from the bottom of the wall, along the ground at the rate of 2 ft. per second; at what rate per second does the other extremity *begin* to descend along the face of the wall ? 7 inches.

19. One end of a ball of thread is fastened to the top of a pole 35 ft. high; a man holding the ball 5 ft. above the ground moves uniformly from the bottom at the rate of five miles an hour, allowing the thread to unwind as he advances. What is the man's distance from the pole when the thread is unwinding at the rate of one mile per hour ?  $\frac{1}{2}\sqrt{6}$  ft.

20. A vessel sailing due south at the uniform rate of 8 miles per hour is 20 miles north of a vessel sailing due east at the rate of 10 miles an

hour. At what rate are they separating—(α) at the end of  $1\frac{1}{2}$  hours? (β) at the end of  $2\frac{1}{2}$  hours?

*Express the distances in terms of the time.* (α)  $5\frac{1}{17}$  miles per hour.

21. When are the two ships mentioned in the preceding example neither receding from nor approaching each other?

*Put the expression for their rate of separation equal to zero.*

When  $t = \frac{1}{2}$  of an hour.

22. Derive, by the method employed in Art. 29 to determine the differential of the square, the result  $d\left(\frac{1}{x}\right) = \frac{c dx}{x^2}$ ,  $c$  being an unknown constant.

## V.

### *The Product.*

33. Let  $x$  and  $y$  denote any two variables; in order to derive the differential of their product, we express  $xy$  by means of squares, since we have already obtained a formula for the differentiation of the square. From the identity

$$(x+y)^2 = x^2 + 2xy + y^2,$$

we derive

$$xy = \frac{1}{2}(x+y)^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2.$$

Differentiating,  $d(xy) = (x+y)(dx+dy) - x dx - y dy,$

therefore,  $d(xy) = y dx + x dy. \dots \dots \dots (c)$

Since  $x$  and  $y$  denote any variables whatever, and  $dx$  and  $dy$  their differentials, we can substitute for  $x$  and  $y$  any variable expressions, and for  $dx$  and  $dy$  the corresponding differentials. Thus,

$$\begin{aligned} d[(1+x^2)\sqrt{(a^2-x^2)}] &= \sqrt{(a^2-x^2)}2x dx - \frac{(1+x^2)x dx}{\sqrt{(a^2-x^2)}} \\ &= \frac{2a^2-3x^2-1}{\sqrt{(a^2-x^2)}} x dx. \end{aligned}$$

**34.** Formula (c) is readily extended to products consisting of any number of factors. Thus let  $x_1, x_2, x_3, \dots, x_p$  denote the product of  $p$  variable factors, then

$$\begin{aligned} d(x_1 x_2 x_3 \dots x_p) &= x_2 x_3 \dots x_p dx_1 + x_1 d(x_2 x_3 \dots x_p) \\ &= x_2 x_3 \dots x_p dx_1 + x_1 x_3 \dots x_p dx_2 + x_1 x_2 d(x_3 \dots x_p) \\ &= x_2 x_3 \dots x_p dx_1 + x_1 x_3 \dots x_p dx_2 + \dots + x_1 x_2 \dots x_{p-1} dx_p. \quad (c') \end{aligned}$$

### *The Reciprocal.*

**35.** The differential of the reciprocal may now be obtained by means of the implicit form of this function.

Denoting the function by  $y$ , we have

$$y = \frac{1}{x} \quad \therefore \quad xy = 1.$$

Differentiating the latter equation by formula (c), we obtain

$$y dx + x dy = 0,$$

whence

$$dy = -\frac{y dx}{x};$$

substituting the value of  $y$ ,

$$d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}. \quad \dots \quad (d)$$

Formula (d) enables us to differentiate any fraction of which the denominator alone is variable; thus,

$$d\left(\frac{a+b}{a+x}\right) = -(a+b) \frac{dx}{(a+x)^2}.$$

*The Quotient.*

**36.** By the term *quotient*, as used in this article, we mean a fraction whose numerator and denominator are both variable. In deriving its differential, the quotient is regarded as the product of its numerator by the reciprocal of its denominator. Thus, applying formulas (c) and (d),

$$\begin{aligned} d\left(\frac{x}{y}\right) &= d\left(x\frac{1}{y}\right) = \frac{1}{y}dx + x d\left(\frac{1}{y}\right) \\ &= \frac{dx}{y} - \frac{xdy}{y^2}, \\ \therefore d\left(\frac{x}{y}\right) &= \frac{ydx - xdy}{y^2}. \dots\dots\dots (e) \end{aligned}$$

It will be noticed that the negative sign belongs to the term which contains the differential of the denominator.

As an illustration of the application of this formula, we have

$$d\left(\frac{2x-a}{x^2+b}\right) = \frac{2(x^2+b) - 2x(2x-a)}{(x^2+b)^2}dx = 2\frac{b+ax-x^2}{(x^2+b)^2}dx.$$

Formula (e) is to be used *only when both terms of the fraction are variable*; for, when the numerator is constant, the fraction is equivalent to the product of a constant and the reciprocal of a variable, and, when the denominator is constant, it is equivalent to the product of a constant by a variable factor. Thus, if it be required to differentiate the fraction  $\frac{x^2+a^2}{ax}$ , the use of formula (e) may be avoided by first making the transformation,

$$\frac{x^2+a^2}{ax} = \frac{x}{a} + \frac{a}{x};$$

since, in this form, one term of each fraction is constant. Hence,

$$d\left(\frac{x^2 + a^2}{ax}\right) = \frac{dx}{a} - \frac{adx}{x^2}.$$

### *The Power.*

**37.** To obtain the differential of the power when the exponent is a positive integer, suppose each of the variables  $x_1, x_2, x_3, \dots, x_p$  in formula (c'), Art. 34, to be replaced by  $x$ . The first member contains  $p$  factors, and the second  $p$  terms; the equation therefore reduces to

$$d(x^p) = p x^{p-1} dx. \quad \dots \dots \dots (1)$$

Next, when the exponent is a fraction, let

$$y = x^{\frac{p}{q}}, \quad \text{then} \quad y^q = x^p;$$

differentiating by (1),  $p$  and  $q$  being positive integers, we have

$$\begin{aligned} q y^{q-1} dy &= p x^{p-1} dx, \\ \text{therefore,} \quad dy &= \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} dx. \end{aligned}$$

Substituting the value of  $y$ ,

$$d\left(x^{\frac{p}{q}}\right) = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{\frac{p}{q}-1}} dx = \frac{p}{q} x^{\frac{p}{q}-1} dx. \quad \dots \dots \dots (2)$$

Again, when the exponent is negative, we have

$$x^{-n} = \frac{1}{x^n}.$$



Differentiating by formula (d), Art. 35, we obtain

$$d(x^{-m}) = -\frac{d(x^m)}{x^{2m}},$$

and, since  $m$  is positive, we have, by (1) or (2),

$$d(x^{-m}) = -\frac{mx^{m-1}dx}{x^{2m}} = -mx^{-m-1}dx. \quad \dots (3)$$

Equations (1), (2), and (3) show that, for all values of  $n$ ,

$$d(x^n) = nx^{n-1}dx. \quad \dots (f)$$

By giving to  $n$  the values 2,  $\frac{1}{2}$ , and  $-1$ , successively, it is readily seen that this more general formula includes formulas (a), (b) and (d).

**38.** It is frequently advantageous to transform a given expression by the use of fractional or negative exponents, and employ formula (f) instead of formulas (b) and (d). Thus,

$$d\left[\frac{1}{(a^2 - 2x^2)^{\frac{1}{2}}}\right] = d(a^2 - 2x^2)^{-\frac{1}{2}} = 8(a^2 - 2x^2)^{-\frac{3}{2}}x dx,$$

$$\text{and } d\left[\frac{1}{\sqrt[3]{a+x}}\right] = d(a+x)^{-\frac{1}{3}} = -\frac{1}{3}(a+x)^{-\frac{4}{3}}dx.$$

When the derivative of a function is required, it may be written at once instead of first writing the differential, since the former differs from the latter only in the omission of the factor  $dx$ , which must necessarily occur in every term. Thus, given

$$y = \frac{x}{\sqrt[3]{1+x^3}} = x(1+x^3)^{-\frac{1}{3}},$$

$$\text{we derive } \frac{dy}{dx} = (1+x^3)^{-\frac{1}{3}} - \frac{1}{3}x(1+x^3)^{-\frac{4}{3}} \cdot 3x = \frac{1}{(1+x^3)^{\frac{4}{3}}}.$$

## Examples V.

1. From the identity  $xy = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2$  derive the formula for differentiating the product.

2. Differentiate  $\frac{a + bx + cx^2}{x}$ .

Put the expression in the form  $\frac{a}{x} + b + cx$ .

$$\left(c - \frac{a}{x^2}\right)dx.$$

3. Find the derivative of

$$y = \frac{a^2 - b^2}{a^2 - x^2}. \quad \text{See remark, Art. 35.}$$

$$\frac{dy}{dx} = (a^2 - b^2) \frac{2x}{(a^2 - x^2)^2}.$$

4.  $y = \sqrt{x^2 - a^2}.$

$$\frac{dy}{dx} = \frac{3x^2}{2\sqrt{x^2 - a^2}}.$$

5.  $y = \frac{2x^4}{a^2 - x^2}.$

$$\frac{dy}{dx} = \frac{4x^3(2a^2 - x^2)}{(a^2 - x^2)^2}.$$

6.  $y = (1 + 2x^2)(1 + 4x^2).$

$$\frac{dy}{dx} = 4x(1 + 3x + 10x^3).$$

7.  $y = (a^2 + x^2)(b^2 + 3x^2).$

$$\frac{dy}{dx} = 3(5x^3 + b^2x + 2a^2)x.$$

8.  $y = (1 + x)^4(1 + x^2).$

$$\frac{dy}{dx} = 4(1 + x)^3(1 + x^2)(1 + x + 2x^2).$$

9.  $y = (1 + x^m)^n + (1 + x^n)^m.$

$$\frac{dy}{dx} = mn[(1 + x^m)^{n-1}x^{m-1} + (1 + x^n)^{m-1}x^{n-1}].$$

10.  $y = \frac{x^2 - 2a^2}{x - a}.$

$$\frac{dy}{dx} = 1 + \frac{a^2}{(x - a)^2}.$$

11.  $y = \frac{a - x}{\sqrt{x}}.$

$$\frac{dy}{dx} = -\frac{a + x}{2x^{\frac{3}{2}}}.$$

$$12. y = \frac{\sqrt[4]{(x^2 - a^2)}}{x}.$$

$$\frac{dy}{dx} = \frac{a^2}{x^2 \sqrt[4]{(x^2 - a^2)}}.$$

$$13. y = \frac{ab}{cx \sqrt{(x^2 - a^2)}}. \text{ See Art. 38.}$$

$$\frac{dy}{dx} = -\frac{ab}{c} \cdot \frac{2x^2 - a^2}{x^3(x^2 - a^2)^{\frac{3}{2}}}.$$

$$14. y = \frac{1}{\sqrt{(1+x)}} + \frac{1}{\sqrt{(1-x)}}.$$

$$\frac{dy}{dx} = \frac{1}{2}[(1-x)^{-\frac{1}{2}} - (1+x)^{-\frac{1}{2}}].$$

$$15. y = (1+x) \sqrt{(1-x)}.$$

$$\frac{dy}{dx} = \frac{1-3x}{2\sqrt{(1-x)}}.$$

$$16. y = (a+x)^2(b-x)^4x^2.$$

$$\frac{dy}{dx} = x(a+x)^2(b-x)^2[2ab + (5b-6a)x - 9x^2].$$

$$17. y = \frac{x^a + 1}{x^a - 1}.$$

$$\frac{dy}{dx} = -\frac{2ax^{a-1}}{(x^a - 1)^2}.$$

$$18. y = (3b + 2ax)^{\frac{1}{2}}(b - ax).$$

$$\frac{dy}{dx} = -5a^2x \sqrt{(3b + 2ax)}.$$

$$19. y = \frac{a^2 - b^2}{(2ax - x^2)^{\frac{1}{2}}}.$$

Put in the form  $(a^2 - b^2)(2ax - x^2)^{-\frac{1}{2}}$ .  $\frac{dy}{dx} = 3(a^2 - b^2) \frac{x - a}{(2ax - x^2)^{\frac{3}{2}}}.$

$$20. y = \frac{x}{\sqrt{(a^2 - x^2)}}.$$

$$\frac{dy}{dx} = \frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

$$21. y = \frac{bx}{\sqrt{(2ax - x^2)}}.$$

$$\frac{dy}{dx} = \frac{abx}{(2ax - x^2)^{\frac{3}{2}}}.$$

$$22. y = \sqrt{\frac{1+x}{1-x}}.$$

$$\frac{dy}{dx} = \frac{1}{(1-x)\sqrt{(1-x^2)}}.$$

$$23. y = \frac{x}{\sqrt{(a^2 + x^2)} - x}.$$

Rationalize the denominator.

$$\frac{dy}{dx} = \frac{1}{a^2} \left[ \frac{a^2 + 2x^2}{\sqrt{(a^2 + x^2)}} + 2x \right].$$

24. Two locomotives are moving along two straight lines of railway which intersect at an angle of  $60^\circ$ ; one is approaching the intersection at the rate of 25 miles an hour, and the other is receding from it at the rate of 30 miles an hour; find the rate per hour at which they are separating from each other when each is 10 miles from the intersection.

$2\frac{1}{2}$  miles.

25. A street-crossing is 10 ft. from a street-lamp situated directly above the curbstone, which is 60 ft. from the vertical walls of the opposite buildings. If a man is walking across to the opposite side of the street at the rate of 4 miles an hour, at what rate per hour does his shadow move upon the walls—(α) when he is 5 ft. from the curbstone? (β) when he is 20 ft. from the curbstone?

(α) 96 miles; (β) 6 miles.

26. Assuming the volume of a tree to be proportional to the cube of its diameter, and that the latter increases uniformly; find the ratio of the rate of its volume when the diameter is 6 inches to the rate when the diameter is 3 ft.

$\frac{1}{8}$ .

27. If an ingot of silver in the form of a parallelopiped expand  $\frac{1}{1000}$  part of each of its linear dimensions for each degree of temperature, at what rate per degree of temperature is its volume increasing when the sides are respectively 2, 3, and 6 inches?

If  $x$  denote a side,  $dx$  may be assumed to denote the rate per degree of temperature.

$\frac{3}{800}$  of a cubic inch.

28. Prove generally that, if the coefficient of expansion of each linear dimension of a solid is  $k$ , its coefficient of expansion in volume is  $3k$ .

*Solution.*—

Let  $x$  denote any side; then, if  $V$  denote the volume, we shall have  $V = cx^3$ ;  $c$  being a constant dependent on the shape of the body.

Therefore

$$dV = 3cx^2 dx;$$

or, since

$$dx = kx,$$

$$dV = 3kcx^3 = 3kV.$$

## CHAPTER III.

### THE DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

#### VI.

#### *The Logarithmic Function.*

**39.** IN this chapter, the formulas for the differentiation of the simple transcendental functions are to be established.

We begin by deducing the differential of the logarithmic function, employing the method exemplified in Art. 29.

The symbol  $\log x$  is used in this article to denote the logarithm of  $x$  to any base, and  $\log_b x$  is used when we wish to designate a particular base  $b$ .

$$\text{Let } z = mx, \quad \therefore \quad \log z = \log m + \log x,$$

differentiating by Art. 21,

$$dz = m dx, \quad \text{and} \quad d(\log z) = d(\log x);$$

whence 
$$\frac{d(\log z)}{dz} = \frac{d(\log x)}{m dx}$$

Multiplying by  $z = mx$ , to eliminate  $m$ , we obtain

$$z \frac{d(\log z)}{dz} = x \frac{d(\log x)}{dx}. \quad \dots \dots (1)$$

The derivatives,  $\frac{d(\log z)}{dz}$  and  $\frac{d(\log x)}{dx}$ , are, by Art. 23, func-

tions of  $s$  and of  $x$  respectively, independent of the values of  $ds$  and  $dx$ ; moreover, equation (1) is true for all values of  $x$  and  $s$ , these quantities being entirely independent of each other, since the arbitrary ratio  $m$  has been eliminated. Hence, in equation (1), one of the quantities,  $x$  or  $s$ , may be assumed to have a fixed value, while the other is variable; whence it follows that the members of this equation have a fixed value independent of the values of  $x$  and  $s$ ; we therefore write

$$x \frac{d(\log x)}{dx} = \text{a constant.} \quad \dots \dots \dots (2)$$

This constant, although independent of  $x$ , may be dependent on the value of the base of the system of logarithms under consideration. Denoting the base of the system by  $b$ , we therefore denote the constant by  $B$ , and write equation (2) thus,—

$$d(\log_b x) = \frac{B dx}{x}. \quad \dots \dots \dots (3)$$

**40.** *To determine the value of  $B$ ,* we establish a relation between two values of the base and the corresponding values of this unknown quantity.

Denoting another value of the base by  $a$ , and the corresponding value of the unknown constant by  $A$ , we have

$$d(\log_a x) = \frac{A dx}{x}. \quad \dots \dots \dots (4)$$

The relation sought may now be obtained by differentiating, by means of (3) and (4), the identical equation

$$\log_a x = \log_a b \log_b x,^* \quad \dots \dots \dots (5)$$

---

\* This identity is most readily obtained thus,—by definition

$$x = b^{\log_b x}.$$

thus obtaining  $\frac{A dx}{x} = \log_a b \frac{B dx}{x},$

or  $B \log_a b = A,$

hence  $\log_a b^B = A,$

that is,  $A$  is the logarithm to the base  $a$  of  $b^B$ ; whence we have

$$b^B = a^A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now, it is obvious that the value of  $a^A$  cannot depend upon  $b$ , hence equation (6) shows that the value of  $b^B$  likewise cannot depend upon  $b$ ;  $b^B$  must, therefore, have a value entirely independent of  $b$ . Denoting this constant value by  $\epsilon$ , we write

$$b^B = \epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Adopting this constant as a base, and taking the logarithms of each member of equation (7), we have

$$B \log_\epsilon b = 1,$$

whence  $B = \frac{1}{\log_\epsilon b}.$

Introducing this value of  $B$  in equation (3), we obtain

$$d(\log_b x) = \frac{dx}{\log_\epsilon b \cdot x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

In this equation, the differential of a logarithm to any given base is expressed by the aid of the unknown constant  $\epsilon$ .

**41.** The constant  $\epsilon$  is employed as the base of a system of

taking the logarithm to the base  $a$  of each member, we have

$$\log_a x = \log_a x \log_a b.$$

logarithms, sometimes called *natural* or *hyperbolic*, but more commonly *Napierian* logarithms, from the name of the inventor of logarithms. Hence  $\epsilon$  is known as the *Napierian base*.

Putting  $b = \epsilon$  in formula (g) we derive

$$d(\log_{\epsilon} x) = \frac{dx}{x} \cdot \dots \cdot (g')$$

The logarithms employed in analytical investigations are almost exclusively Napierian. Whenever it is necessary, for the purpose of obtaining numerical results, these logarithms may be expressed in terms of the common tabular logarithms by means of the formula,

$$\log_{10} x = \log_{10} \epsilon \log_{\epsilon} x,$$

which is derived from equation (5), Art. 40, by writing 10 for  $a$  and  $\epsilon$  for  $b$ . The value of the constant  $\log_{10} \epsilon$  will be computed in a subsequent chapter.

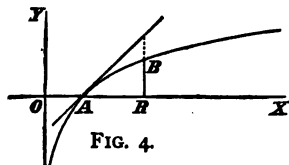
Hereafter, whenever the symbol  $\log$  is employed without the subscript,  $\log_{\epsilon}$  is to be understood.

### *The Logarithmic Curve.*

**42.** The curve, corresponding to the equation

$$y = \log_{\epsilon} x \cdot \dots \cdot (1)$$

is called the *logarithmic curve*.



The shape of this curve is indicated in Fig. 4. It passes through the point  $A$  whose coordinates are  $(1, 0)$ , since

$$\log 1 = 0.$$

Since we have, from formula (g'),



$$\tan \phi = \frac{dy}{dx} = \frac{1}{x}, \quad . . . . . (2)$$

the value of  $\tan \phi$  at the point  $A$  is unity, and therefore the tangent line at this point cuts the axis of  $x$  at an angle of  $45^\circ$ , as in the diagram. We have from equation (2),

when  $x > 1$   $\tan \phi < 1$ ,  
and when  $x < 1$   $\tan \phi > 1$ ;

the curve, therefore, lies below this tangent, as shown in Fig. 4.

The point  $(\epsilon, 1)$  is a point of the curve; let  $B$ , Fig. 4, be this point, then  $OR$  will represent the Napierian base, and  $BR = 1$ . Since

$$OA = 1, \quad \text{and} \quad AR > BR,$$

$$OR > 2;$$

that is, the Napierian base  $\epsilon$  is somewhat greater than 2.

The quantity  $\epsilon$  is incommensurable: the method of computing its value to any required degree of accuracy is given in a subsequent chapter.

### *Logarithmic Differentiation.*

**43.** The differential of the Napierian logarithm of the variable  $x$ , that is the expression  $\frac{dx}{x}$ , is called the *logarithmic differential* of  $x$ .

When  $x$  has a negative value, the expression  $\log x$  has no real value; in this case, however,  $\log(-x)$  is real, and we have

$$d[\log(-x)] = \frac{d(-x)}{-x} = \frac{dx}{x}.$$

This expression therefore, in the case of a negative quantity, is identical with the logarithmic differential of the positive quantity having the same numerical value.

**44.** The process of taking logarithms and differentiating the result is called *logarithmic differentiation*. By means of this method, all the formulas for the differentiation of algebraic functions may be derived.

In the following logarithmic equations, it is to be understood that that sign is taken in each case which will render the logarithm real.

By differentiating the formulas,—

$$\log (\pm xy) = \log (\pm x) + \log (\pm y),$$

$$\log \left( \pm \frac{x}{y} \right) = \log (\pm x) - \log (\pm y),$$

$$\log (\pm x^n) = n \log (\pm x),$$

we obtain 
$$\frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y},$$

$$\frac{y}{x} d\left(\frac{x}{y}\right) = \frac{dx}{x} - \frac{dy}{y},$$

$$\frac{d(x^n)}{x^n} = n \frac{dx}{x}.$$

These formulas are evidently equivalent to (c), (e), and (f), of which we thus have an independent proof.

**45.** The method of logarithmic differentiation may frequently be used with advantage in finding the derivatives of complicated algebraic expressions. For example, let us take

$$u = \frac{\sqrt{(2x)(1-x^2)^{\frac{1}{2}}}}{(x-2)^{\frac{1}{2}}}. \quad \dots \dots \dots (1)$$

Hence, we derive

$\log u = \frac{1}{2} \log (2x) + \frac{1}{4} \log (1 - x^2) - \frac{1}{3} \log (x - 2), \dots (2)$   
differentiating,

$$\frac{du}{u dx} = \frac{1}{2x} - \frac{1}{2} \frac{x}{1 - x^2} - \frac{1}{3} \frac{1}{x - 2}, \dots (3)$$

adding and reducing,

$$\frac{du}{u dx} = \frac{-8x^3 + 24x^2 - x - 6}{6(1 - x^2)(x - 2)x};$$

therefore 
$$\frac{du}{dx} = \frac{-8x^3 + 24x^2 - x - 6}{3(2x)^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}(x - 2)^{\frac{1}{2}}}.$$

For certain values of  $x$ , one or more of the quantities whose logarithms appear in equation (2) become negative. When this is the case these logarithms should, strictly speaking, be replaced by the logarithms of the numerical values of the quantities in question; this change however would not affect the form of equation (3). See Art. 43.

### *Exponential Functions.*

**46.** An *exponential function* is an expression in which an exponent is a function of the independent variable. The quantity affected by the exponent may be constant or variable. In the first case, let the function be denoted by

$$y = a^x. \dots (1)$$

If  $a$  is negative,  $a^x$  cannot denote a continuously varying quantity. We therefore exclude the case in which  $a$  has a negative value, and regard  $a^x$  as a continuously varying positive quantity.

Taking Napierian logarithms of both members of equation (1), we have

$$\log y = x \log a;$$

differentiating by ( $g'$ ),

$$\frac{dy}{y} = \log a \cdot dx;$$

hence

$$dy = \log a \cdot y dx,$$

or

$$d(a^x) = \log a \cdot a^x dx. \quad \dots \dots \dots (h)$$

Exponential functions of the form  $\epsilon^x$  are of frequent occurrence. Putting  $a = \epsilon$  in formula (h), we have

$$d(\epsilon^x) = \epsilon^x dx; \quad \dots \dots \dots (h')$$

hence the derivative of the function  $\epsilon^x$  is identical with the function itself. This function is the inverse of the Napierian logarithm; it has been proposed to denote it by the symbol  $\exp x$ .

**47.** When both the exponent and the quantity affected by it are variable, the method of logarithmic differentiation may be employed. Thus, if the given function be

$$z = (nx)^{x^2},$$

we shall have

$$\log z = x^2 \log (nx);$$

differentiating,

$$\frac{dz}{z} = x^2 \frac{dx}{x} + 2x \log (nx) dx,$$

hence

$$d[(nx)^{x^2}] = (nx)^{x^2} x [1 + 2 \log (nx)] dx.$$

### Examples VI.

1. Given the function  $y = \log_{\epsilon} x$ ; show that  $\left. \frac{dy}{dx} \right|_{\epsilon} = \frac{\log_{\epsilon} \epsilon}{\epsilon}$ , and hence prove that the tangent to the corresponding curve, at the point whose abscissa is  $\epsilon$ , passes through the origin.

Put  $a = x = \epsilon$  in equation 5, Art. 40.

$$2. y = x^n \log x. \quad \frac{dy}{dx} = x^{n-1}(1 + n \log x).$$

$$3. y = \log(\log x). \quad \frac{dy}{dx} = \frac{1}{x \log x}.$$

$$4. y = \log[\log(a + bx^n)]. \quad \frac{dy}{dx} = \frac{nbx^{n-1}}{(a + bx^n) \log(a + bx^n)}.$$

$$5. y = \sqrt{x} - \log(\sqrt{x} + 1). \quad \frac{dy}{dx} = \frac{1}{2(\sqrt{x} + 1)}.$$

$$6. y = \log \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}. \quad \frac{dy}{dx} = \frac{\sqrt{a}}{(a - x)\sqrt{x}}.$$

Put in the form,  $\log(\sqrt{a} + \sqrt{x}) - \log(\sqrt{a} - \sqrt{x})$ .

$$7. y = \log[\sqrt{x-a} + \sqrt{x-b}]. \quad \frac{dy}{dx} = \frac{1}{2\sqrt{(x-a)(x-b)}}.$$

$$8. y = \log[x + \sqrt{x^2 \pm a^2}]. \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2 \pm a^2}}.$$

$$9. y = \log \frac{x}{\sqrt{1+x^2}}. \quad \frac{dy}{dx} = \frac{1}{x(1+x^2)}.$$

$$10. y = \log \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}. \quad \frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}.$$

$$11. y = \log[x + \sqrt{a^2 - x^2}]. \quad \frac{dy}{dx} = \frac{\sqrt{a^2 - x^2} - x}{\sqrt{a^2 - x^2}[x + \sqrt{a^2 - x^2}]}$$

$$12. y = \log \frac{x}{\sqrt{x^2 + a^2} - x}. \quad \frac{dy}{dx} = \frac{1}{x} + \frac{1}{\sqrt{x^2 + a^2}}.$$

$$13. y = \log[\sqrt{1+x^2} + \sqrt{1-x^2}]. \quad \frac{dy}{dx} = \frac{1}{x} - \frac{1}{x\sqrt{1-x^2}}.$$

$$14. y = \log(x-a) - \frac{a(2x-a)}{(x-a)^2}. \quad \frac{dy}{dx} = \frac{x^2 + a^2}{(x-a)^2}.$$

15.  $y = a^{2x}$ .

$$\frac{dy}{dx} = 2 \log a \cdot a^{2x} x.$$

16.  $y = e^{\frac{1}{1+x}}$ .

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2} \cdot e^{\frac{1}{1+x}}.$$

17.  $y = e^x(1-x^2)$ .

$$\frac{dy}{dx} = e^x(1-3x^2-x^2).$$

18.  $y = (x-3)e^{2x} + 4xe^x$ .

$$\frac{dy}{dx} = (2x-5)e^{2x} + 4(x+1)e^x.$$

19.  $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ .

$$\frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}.$$

20.  $y = b^{a^x}$ .

$$\frac{dy}{dx} = \log a \cdot \log b \cdot b^{a^x} \cdot a^x.$$

21.  $y = a^{x^n}$ .

$$\frac{dy}{dx} = n a^{x^n} \cdot x^{n-1} \cdot \log a.$$

22.  $y = \frac{x}{e^x - 1}$ .

$$\frac{dy}{dx} = \frac{e^x(1-x) - 1}{(e^x - 1)^2}.$$

23.  $y = \log(e^x + e^{-x})$ .

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

24.  $y = a^{\log a}$ .

$$\frac{dy}{dx} = \frac{1}{x} \log_e a \cdot a^{\log a}.$$

25.  $y = \log \frac{e^x}{1+e^x}$ .

$$\frac{dy}{dx} = \frac{1}{1+e^x}.$$

26.  $y = x^x$ .

$$\frac{dy}{dx} = x^x(1 + \log x).$$

27.  $y = \frac{(x-1)^{\frac{1}{2}}}{(x-2)^{\frac{1}{2}}(x-3)^{\frac{1}{2}}}$ .

$$\frac{dy}{dx} = -\frac{(x-1)^{\frac{1}{2}}(7x^2+30x-97)}{12(x-2)^{\frac{3}{2}}(x-3)^{\frac{3}{2}}}.$$

See Art. 45.

$$28. y = \frac{\sqrt{[ax(x-3a)]}}{\sqrt{(x-4a)}}.$$

$$\frac{dy}{dx} = \frac{\sqrt{a(x^2 - 8ax + 12a^2)}}{2[x(x-3a)]^{\frac{1}{2}}(x-4a)^{\frac{3}{2}}}.$$

$$29. y = \frac{(x+1)^{\frac{1}{2}}(x+3)^{\frac{2}{3}}}{(x+2)^4}.$$

$$\frac{dy}{dx} = \frac{x^2(x+3)^{\frac{1}{2}}}{(x+2)^5(x-1)^{\frac{1}{2}}}.$$

## VII.

*The Trigonometric or Circular Functions.*

**48.** In deriving the differentials of the trigonometric functions of a variable angle, we employ the *circular measure* of the angle, and denote it by  $\theta$ . Thus, let  $s$  denote the length of the arc subtending the angle in the circle whose radius is  $a$ , then

$$\theta = \frac{s}{a}.$$

In Fig. 5, let  $OA$  be a fixed line, and  $OP$  an equal line rotating about the origin  $O$ ; then  $P$  will describe the circle whose equation (the coordinates being rectangular) is

$$x^2 + y^2 = a^2.$$

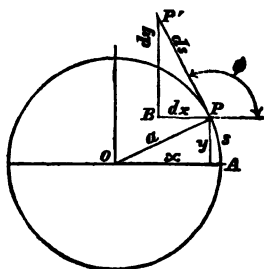


FIG. 5.

The velocity of the point  $P$  is the rate of  $s$ , and (see Art. 17) is denoted by  $\frac{ds}{dt}$ , which has a positive value when  $P$  moves so as to increase  $\theta$ . Let  $PP'$ ,

taken in the direction of the motion of  $P$ , represent  $ds$ ; then, according to the definition given in Art. 25,  $PP'$  is a tangent line, and  $PB$  and  $BP'$  will represent  $dx$  and  $dy$ , as in Art. 26.

**49.** We have first to show that the line  $PP'$ , which is a tangent to the curve according to the general definition (Art. 25), is perpendicular to the radius.

Differentiating the equation of the circle, we have

$$x dx + y dy = 0;$$

whence 
$$\tan \phi = \frac{dy}{dx} = -\frac{x}{y}.$$

Now (see Fig. 5), 
$$\frac{y}{x} = \tan \theta,$$

therefore, 
$$\tan \phi = -\cot \theta = \tan(\theta \pm \tfrac{1}{2}\pi),$$

or, 
$$\phi = \theta \pm \tfrac{1}{2}\pi;$$

hence the tangent line is perpendicular to the radius. Assuming  $\phi$  to be the angle between the positive directions of  $x$  and  $ds$ , we have

$$\phi = \theta + \tfrac{1}{2}\pi.$$

### *The Sine and the Cosine.*

**50.** From Fig. 5, it is evident that

$$\sin \theta = \frac{y}{a}, \quad \text{and} \quad \cos \theta = \frac{x}{a},$$

therefore 
$$d(\sin \theta) = \frac{dy}{a}, \quad \text{and} \quad d(\cos \theta) = \frac{dx}{a}. \quad \dots \quad (1)$$

In equations (1) we have to express  $dy$  and  $dx$  in terms of  $\theta$  and  $d\theta$ .



Again, from the figure, we have

$$dy = \sin \phi \cdot ds, \quad \text{and} \quad dx = \cos \phi \cdot ds;^*$$

substituting in equations (1), we obtain

$$d(\sin \theta) = \sin \phi \frac{ds}{a}, \quad \text{and} \quad d(\cos \theta) = \cos \phi \frac{ds}{a}. \quad \dots (2)$$

Since  $\phi = \theta + \frac{1}{2} \pi$ , and  $\frac{s}{a} = \theta$ ,

$$\sin \phi = \cos \theta, \quad \cos \phi = -\sin \theta, \quad \text{and} \quad \frac{ds}{a} = d\theta.$$

Substituting these values in equations (2), we obtain

$$d(\sin \theta) = \cos \theta d\theta, \quad \dots (i)$$

and  $d(\cos \theta) = -\sin \theta d\theta. \dots (j)$

### *The Tangent and the Cotangent.*

51. The differential of  $\tan \theta$  is found by applying formula (e) to the equation

$$\tan \theta = \frac{\sin \theta}{\cos \theta};$$

thus, 
$$d(\tan \theta) = \frac{\cos \theta d(\sin \theta) - \sin \theta d(\cos \theta)}{\cos^2 \theta},$$

or 
$$d(\tan \theta) = \frac{d\theta}{\cos^2 \theta} = \sec^2 \theta d\theta. \quad \dots (k)$$

---

\*In Fig. 5,  $dx$  is negative; but,  $\phi$  being in the second quadrant,  $\cos \phi$  is likewise negative.

The differential of  $\cot \theta$  is found by applying formula (k) to the equation

$$\cot \theta = \tan \left( \frac{1}{2} \pi - \theta \right);$$

whence  $d(\cot \theta) = -\frac{d\theta}{\sin^2 \theta} = -\operatorname{cosec}^2 \theta d\theta. \quad \dots (l)$

### *The Secant and the Cosecant.*

**52.** The differential of  $\sec \theta$  is found by applying formula (d) to the equation

$$\sec \theta = \frac{1}{\cos \theta};$$

whence  $d(\sec \theta) = \frac{\sin \theta d\theta}{\cos^2 \theta} = \sec \theta \tan \theta d\theta. \quad \dots (m)$

The differential of  $\operatorname{cosec} \theta$  is found by applying formula (m) to the equation

$$\operatorname{cosec} \theta = \sec \left( \frac{1}{2} \pi - \theta \right);$$

whence  $d(\operatorname{cosec} \theta) = -\frac{\cos \theta d\theta}{\sin^2 \theta} = -\operatorname{cosec} \theta \cot \theta d\theta. \quad \dots (n)$

### *The Versed-Sine.*

**53.** The *versed-sine* is defined by the equation

$$\operatorname{vers} \theta = 1 - \cos \theta;$$

therefore  $d(\operatorname{vers} \theta) = \sin \theta d\theta. \quad \dots (o)$

**Examples VII.**

1. The value of  $d(\sin \theta)$  being given, derive that of  $d(\cos \theta)$  from the formula

$$\cos \theta = \sin \left( \frac{1}{2} \pi - \theta \right);$$

also from the identity

$$\cos^2 \theta = 1 - \sin^2 \theta.$$

2. From the identity  $\sec^2 \theta = 1 + \tan^2 \theta$ , derive the differential of  $\sec \theta$ .

3. From the identity  $\sin 2\theta = 2 \sin \theta \cos \theta$ , derive another by taking derivatives.

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta.$$

4. From the identity  $\sin \left( \theta \pm \frac{1}{2} \pi \right) = \frac{1}{2} \sqrt{2} (\sin \theta \pm \cos \theta)$ , derive another by taking derivatives.

$$\cos \left( \theta \pm \frac{1}{2} \pi \right) = \frac{1}{2} \sqrt{2} (\cos \theta \mp \sin \theta).$$

5. Prove the formulas :—

$$d(\log \sin \theta) = -d(\log \operatorname{cosec} \theta) = \cot \theta d\theta;$$

$$d(\log \cos \theta) = -d(\log \sec \theta) = -\tan \theta d\theta;$$

$$d(\log \tan \theta) = -d(\log \cot \theta) = (\tan \theta + \cot \theta) d\theta.$$

6. Obtain an identity by taking derivatives of both members of the equation

$$\tan \frac{1}{2} \theta = \frac{1 - \cos \theta}{\sin \theta}.$$

$$\frac{1}{2} \sec^2 \frac{1}{2} \theta = \frac{1 - \cos \theta}{\sin^2 \theta}.$$

$$7. y = \theta + \sin \theta \cos \theta.$$

$$\frac{dy}{d\theta} = 2 \cos^2 \theta$$

$$8. y = \sin \theta - \frac{1}{2} \sin^3 \theta.$$

$$\frac{dy}{d\theta} = \cos^3 \theta.$$

$$9. y = \frac{\sin \theta}{\sqrt{(\cos \theta)}}.$$

$$\frac{dy}{d\theta} = \frac{1 + \cos^2 \theta}{2 (\cos \theta)^{\frac{3}{2}}}.$$

$$10. y = \frac{1}{3} \tan^3 \theta - \tan \theta + \theta.$$

$$\frac{dy}{d\theta} = \tan^4 \theta.$$

$$11. y = \frac{1}{3} \tan^3 \theta + \tan \theta.$$

$$\frac{dy}{d\theta} = \sec^4 \theta.$$

$$12. y = \sin e^x.$$

$$\frac{dy}{dx} = e^x \cos e^x.$$

$$13. y = x \sin x^2.$$

$$\frac{dy}{dx} = \sin x^2 + 2x^2 \cos x^2.$$

$$14. y = a^{\sin x}.$$

$$\frac{dy}{dx} = \log a \cdot a^{\sin x} \cos x.$$

$$15. y = \tan^2 \theta + \log (\cos^2 \theta).$$

$$\frac{dy}{d\theta} = 2 \tan^2 \theta.$$

$$16. y = \log (\tan \theta + \sec \theta).$$

$$\frac{dy}{d\theta} = \sec \theta.$$

$$17. y = \log \tan \left( \frac{1}{2} \pi + \frac{1}{2} \theta \right).$$

$$\frac{dy}{d\theta} = \frac{1}{\cos \theta}.$$

$$18. y = x + \log \cos \left( \frac{1}{2} \pi - x \right).$$

$$\frac{dy}{dx} = \frac{2}{1 + \tan x}.$$

$$19. y = \log \sqrt[4]{\sin x} + \log \sqrt[4]{\cos x}.$$

$$\frac{dy}{dx} = \cot 2x.$$

$$20. y = \sin n \theta (\sin \theta)^n.$$

$$\frac{dy}{d\theta} = n (\sin \theta)^{n-1} \sin (n+1) \theta.$$

$$21. y = \frac{\sin x}{1 + \tan x}.$$

$$\frac{dy}{dx} = \frac{\cos^2 x - \sin^2 x}{(\sin x + \cos x)^2}.$$

$$22. y = e^{ax} \cos bx.$$

$$\frac{dy}{dx} = e^{ax} (a \cos bx - b \sin bx).$$

$$23. y = \log \sqrt{\frac{a \cos x - b \sin x}{a \cos x + b \sin x}}.$$

$$\frac{dy}{dx} = \frac{-ab}{a^2 \cos^2 x - b^2 \sin^2 x}.$$

$$24. y = e^x (\cos x - \sin x). \quad \frac{dy}{dx} = -2e^x \sin x.$$

25. The crank of a small steam-engine is 1 foot in length, and revolves uniformly at the rate of two turns per second, the connecting rod being 5 ft. in length; find the velocity per second of the piston when the crank makes an angle of  $45^\circ$  with the line of motion of the piston-rod; also when the angle is  $135^\circ$ , and when it is  $90^\circ$ .

*Solution* :—

Let  $a$ ,  $b$ , and  $x$  denote respectively the crank, the connecting-rod, and the variable side of the triangle; and let  $\theta$  denote the angle between  $a$  and  $x$ .

We easily deduce

$$x = a \cos \theta + \sqrt{b^2 - a^2 \sin^2 \theta};$$

$$\text{whence} \quad \frac{dx}{dt} = - \left( a \sin \theta + \frac{a^2 \sin \theta \cos \theta}{\sqrt{b^2 - a^2 \sin^2 \theta}} \right) \frac{d\theta}{dt}.$$

In this case,  $\frac{d\theta}{dt} = 4\pi$ ,  $a = 1$ , and  $b = 5$ .

$$\text{When } \theta = 45^\circ, \frac{dx}{dt} = -\frac{16\pi\sqrt{2}}{7} \text{ ft.}$$

26. An elliptical cam revolves at the rate of two turns per second about a horizontal axis passing through one of the foci, and gives a reciprocating motion to a bar moving in vertical guides in a line with the centre of rotation: denoting by  $\theta$  the angle between the vertical and the major axis, find the velocity per second with which the bar is moving when  $\theta = 60^\circ$ , the eccentricity of the ellipse being  $\frac{1}{2}$ , and the semi-major axis 9 inches. Also find the velocity when  $\theta = 90^\circ$ .

*The relation between  $\theta$  and the radius vector is expressed by the equation*

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}.$$

$$\text{When } \theta = 60^\circ, \frac{dr}{dt} = -12\sqrt{3}\pi \text{ inches.}$$

27. Find an expression in terms of its azimuth for the rate at which the altitude of a star is increasing.

*Solution* :—

Let  $h$  denote the altitude and  $A$  the azimuth of the star,  $\rho$  its polar distance,  $t$  the hour angle, and  $L$  the latitude of the observer; the formulas of spherical trigonometry give

$$\sin h = \sin L \cos p + \cos L \sin p \cos t, \quad . . . . (1)$$

and  $\sin p \sin t = \sin A \cos h. \quad . . . . (2)$

Differentiating (1),  $p$  and  $L$  being constant,

$$\cos h \frac{dh}{dt} = - \cos L \sin p \sin t,$$

whence, substituting the value of  $\sin p \sin t$ , from equation (2),

$$\frac{dh}{dt} = - \cos L \sin A.$$

It follows that  $\frac{dh}{dt}$  is greatest when  $\sin A$  is numerically greatest; that is, when the star is on the prime vertical. In the case of a star that never reaches the prime vertical, the rate is greatest when  $A$  is greatest.

## VIII.

### *The Inverse Circular Functions.*

54. It is shown in Trigonometry that, if

$$x = \sin \theta,$$

the expressions

$$2n\pi + \theta \quad \text{and} \quad (2n+1)\pi - \theta, \quad . . (1)$$

in which  $n$  denotes zero or any integer, include all the arcs of which the sine is  $x$ ; hence each of these arcs is a value of the *inverse function*

$$\sin^{-1} x.$$

Among these values, there is always one, and *only one*, which falls between  $-\frac{1}{2}\pi$  and  $+\frac{1}{2}\pi$ ; since, while the arc

passes from the former of these values to the latter, the sine passes from  $-1$  to  $+1$ ; that is, it passes once through all its possible values.

Let  $\theta$ , in the expressions (1), denote this value, which we shall call the *primary* value of the function.

**55.** In a similar manner, if

$$x = \cos \theta,$$

each of the arcs included in the expression

$$2n\pi \pm \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is a value of the inverse function

$$\cos^{-1} x.$$

One of these values, and only one, falls between  $0$  and  $\pi$ ; since, while the arc passes from the former of these values to the latter, its cosine passes from  $+1$  to  $-1$ ; that is, once through all its possible values. In expression (2), let  $\theta$  denote this value, which we shall call the *primary* value of this function.

**56.** In the case of the function

$$\operatorname{cosec}^{-1} x,$$

the definition of the *primary* value that was adopted in the case of  $\sin^{-1} x$ , and the same general expressions (1) for the values of the function, are applicable.

In the case of the function

$$\sec^{-1} x,$$

the definition of the *primary* value adopted in the case of

$\cos^{-1}x$  and expression (2) for the general value of the function are applicable.

Finally, in the case of each of the functions

$$\tan^{-1}x \qquad \text{and} \qquad \cot^{-1}x$$

the *primary* value ( $\theta$ ) is taken between  $-\frac{1}{2}\pi$  and  $+\frac{1}{2}\pi$ , and the general expression for the value of the function is

$$n\pi + \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

### *The Inverse Sine and the Inverse Cosine.*

**57.** To find the differential of the inverse sine, let

$$\theta = \sin^{-1}x;$$

$$\text{then} \quad x = \sin \theta, \quad \text{and} \quad dx = \cos \theta d\theta,$$

$$\text{or} \quad d\theta = \frac{dx}{\cos \theta}.$$

$$\text{Now,} \quad \cos \theta = \pm \sqrt{1 - \sin^2 \theta} = \pm \sqrt{1 - x^2},$$

$$\text{hence} \quad d(\sin^{-1}x) = \frac{dx}{\pm \sqrt{1 - x^2}}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If  $\theta$  denotes the primary value of this function; that is, the value between  $-\frac{1}{2}\pi$  and  $+\frac{1}{2}\pi$ ,  $\cos \theta$  is positive. Hence the upper sign in this ambiguous result belongs to the differential of the primary value of the function; it is therefore usual to write

$$d(\sin^{-1}x) = \frac{dx}{\sqrt{1 - x^2}}. \quad . \quad . \quad . \quad . \quad . \quad (\rho)$$

Since we have, from expressions (1), Art. 54,

$$d(2n\pi + \theta) = d\theta, \quad \text{and} \quad d[(2n+1)\pi - \theta] = -d\theta,$$



it is evident that the positive sign in equation (1) belongs not only to the differential of the primary value of  $\sin^{-1}x$ , but likewise to the differentials of all the values included in  $2n\pi + \theta$ ; and that the negative sign belongs to the differentials of the values of  $\sin^{-1}x$  included in  $(2n+1)\pi - \theta$ .

58. Similarly, if

$$\theta = \cos^{-1}x, \quad x = \cos \theta;$$

whence 
$$d\theta = \frac{dx}{-\sin \theta},$$

or 
$$d(\cos^{-1}x) = \frac{dx}{\mp \sqrt{1-x^2}} \quad . \quad . \quad . \quad (1)$$

If  $\theta$  denote the primary value of the function which in this case is between 0 and  $\pi$ ,  $\sin \theta$  is positive; hence the upper sign in this ambiguous result belongs to the differential of the primary value. It is therefore usual to write

$$d(\cos^{-1}x) = \frac{dx}{-\sqrt{1-x^2}} \quad . \quad . \quad . \quad (2)$$

Since, from expression (2), Art. 55, we have

$$d(2n\pi \pm \theta) = \pm d\theta;$$

it is evident that the upper and lower signs in equation (1) correspond to the upper and lower signs, respectively, in the general expression  $2n\pi \pm \theta$ .

### *The Inverse Tangent and the Inverse Cotangent.*

59. Let

$$\theta = \tan^{-1}x, \quad \text{then} \quad x = \tan \theta;$$

differentiating, we derive,

$$d\theta = \frac{dx}{\sec^2 \theta}.$$

But  $\sec^2 \theta = 1 + \tan^2 \theta = 1 + x^2$ , therefore,

$$d(\tan^{-1} x) = \frac{dx}{1+x^2} \cdot \cdot \cdot \cdot \cdot (r)$$

No ambiguity arises in the value of the differential of this function; since, from expression (3), Art. 56, we have

$$d(n\pi + \theta) = d\theta.$$

Similarly, putting

$$\theta = \cot^{-1} x,$$

we derive

$$d(\cot^{-1} x) = -\frac{dx}{1+x^2} \cdot \cdot \cdot \cdot \cdot (s)$$

### *The Inverse Secant and Inverse Cosecant.*

60. Let

$$\theta = \sec^{-1} x, \quad \text{then} \quad x = \sec \theta;$$

differentiating, we derive

$$d\theta = \frac{dx}{\sec \theta \tan \theta}.$$

But  $\sec \theta = x$ , and  $\tan \theta = \pm \sqrt{(\sec^2 \theta - 1)} = \pm \sqrt{(x^2 - 1)}$ , therefore,

$$d(\sec^{-1} x) = \frac{dx}{\pm x \sqrt{(x^2 - 1)}}.$$

If  $x$  is positive, and if  $\theta$  denotes the primary value of the function,  $\tan \theta$  is positive. Hence it is usual to write

$$d(\sec^{-1} x) = \frac{dx}{x \sqrt{(x^2 - 1)}} \cdot \cdot \cdot \cdot \cdot (t)$$

When  $x$  is negative, if  $\theta$  denotes the primary value of the function, which in this case is in the second quadrant,  $\tan \theta$  is negative; consequently the radical must be taken with the negative sign. Hence, since  $x$  is also negative, the value of the differential is positive, when the arc is taken in the second quadrant.

In like manner we derive

$$d(\operatorname{cosec}^{-1}x) = -\frac{dx}{x\sqrt{x^2-1}} \quad \dots \quad (u)$$

Similar remarks apply also to this differential when  $x$  is negative.

### *The Inverse Versed-Sine.*

61. Let

$$\theta = \operatorname{vers}^{-1}x, \quad \text{then} \quad x = \operatorname{vers} \theta = 1 - \cos \theta,$$

$$\text{and} \quad 1 - x = \cos \theta, \quad \therefore \quad d\theta = \frac{dx}{\sin \theta}.$$

But  $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{2x - x^2}$ , therefore,

$$d(\operatorname{vers}^{-1}x) = \frac{dx}{\sqrt{2x - x^2}} \quad \dots \quad (v)$$

### *Illustrative Examples.*

62. It is sometimes advantageous to transform a given function before differentiating, by means of one of the following formulas:—

$$\sin^{-1} \frac{\alpha}{\beta} = \operatorname{cosec}^{-1} \frac{\beta}{\alpha}, \quad \cos^{-1} \frac{\alpha}{\beta} = \sec^{-1} \frac{\beta}{\alpha}, \quad \tan^{-1} \frac{\alpha}{\beta} = \cot^{-1} \frac{\beta}{\alpha}.$$

Thus, let

$$y = \tan^{-1} \frac{\varepsilon^x \cos x}{1 + \varepsilon^x \sin x},$$

then

$$y = \cot^{-1} (\varepsilon^{-x} \sec x + \tan x).$$

By formula (s),

$$\frac{dy}{dx} = - \frac{\varepsilon^{-x} \sec x \tan x - \varepsilon^{-x} \sec x + \sec^2 x}{\sec^2 x + 2 \varepsilon^{-x} \sec x \tan x + \varepsilon^{-2x} \sec^2 x},$$

multiplying both terms by  $\varepsilon^{2x} \cos^2 x$ ,

$$\frac{dy}{dx} = \frac{\varepsilon^x (\cos x - \sin x - \varepsilon^x)}{1 + 2 \varepsilon^x \sin x + \varepsilon^{2x}}.$$

**63.** Trigonometric substitutions may sometimes be employed with advantage. Thus, let

$$y = \tan^{-1} \frac{x}{\sqrt{(1+x^2)} + 1}.$$

If in this example we put  $x = \tan \theta$ , we have

$$\begin{aligned} y &= \tan^{-1} \frac{\tan \theta}{\sec \theta + 1} = \tan^{-1} \frac{\sin \theta}{1 + \cos \theta} \\ &= \tan^{-1} (\tan \tfrac{1}{2} \theta) = \tfrac{1}{2} \theta = \tfrac{1}{2} \tan^{-1} x. \end{aligned}$$

### Examples VIII.

1. Derive from (p), (r), and (t) the formulas:—

$$d\left(\sin^{-1} \frac{x}{a}\right) = \frac{dx}{\sqrt{a^2 - x^2}};$$

$$d\left(\tan^{-1} \frac{x}{a}\right) = \frac{a dx}{a^2 + x^2};$$

$$d\left(\sec^{-1}\frac{x}{a}\right) = \frac{a \, dx}{x \sqrt{x^2 - a^2}}.$$

2. Derive  $d(\sec^{-1}x)$  from the equation  $\sec^{-1}x = \cos^{-1}\frac{1}{x}$ .

3. Derive  $d\left(\cot^{-1}\frac{x}{a}\right)$  from the equation  $\cot^{-1}\frac{x}{a} = \tan^{-1}\frac{a}{x}$ .

4.  $y = \sin^{-1}(2x^2).$   $\frac{dy}{dx} = \frac{4x}{\sqrt{1-4x^4}}.$

5.  $y = \sin^{-1}(\cos x).$   $\frac{dy}{dx} = -1.$

6.  $y = \sin(\cos^{-1}x).$   $\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}.$

7.  $y = \sin^{-1}(\tan x).$   $\frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}.$

8.  $y = \cos^{-1}(2\cos x).$   $\frac{dy}{dx} = -\frac{2\sin x}{\sqrt{1-4\cos^2 x}}.$

9.  $y = x \sin^{-1}x + \sqrt{1-x^2}.$   $\frac{dy}{dx} = \sin^{-1}x.$

10.  $y = \tan^{-1}e^x.$   $\frac{dy}{dx} = \frac{1}{e^x + e^{-x}}.$

11.  $y = (x^2 + 1) \tan^{-1}x - x.$   $\frac{dy}{dx} = 2x \tan^{-1}x.$

12.  $y = a^2 \sin^{-1}\frac{x}{a} + x \sqrt{a^2 - x^2}.$   $\frac{dy}{dx} = 2 \sqrt{a^2 - x^2}.$

13.  $y = \tan^{-1}\frac{mx}{1-x^2}.$   $\frac{dy}{dx} = \frac{m(1+x^2)}{1+(m^2-2)x^2+x^4}.$

14.  $y = \sin^{-1}\frac{x+1}{\sqrt{2}}.$   $\frac{dy}{dx} = \frac{1}{\sqrt{1-2x-x^2}}.$

$$15. y = \tan^{-1} \frac{x}{\sqrt[4]{(1-x^2)}}. \quad \frac{dy}{dx} = \frac{1}{\sqrt[4]{(1-x^2)}}.$$

$$16. y = \sec^{-1} \frac{a}{\sqrt[4]{(a^2-x^2)}}. \quad \frac{dy}{dx} = \frac{1}{\sqrt[4]{(a^2-x^2)}}.$$

$$17. y = \sin^{-1} \frac{x}{\sqrt[4]{(x^2+a^2)}}. \quad \frac{dy}{dx} = \frac{a}{a^2+x^2}.$$

$$18. y = \sin^{-1} \sqrt[4]{(\sin x)}. \quad \frac{dy}{dx} = \frac{1}{4} \sqrt[4]{(1 + \operatorname{cosec} x)}.$$

$$19. y = \sqrt[4]{(1-x^2)} \sin^{-1} x - x. \quad \frac{dy}{dx} = -\frac{x \sin^{-1} x}{\sqrt[4]{(1-x^2)}}.$$

$$20. y = \tan^{-1} \frac{m+x}{1-mx}. \quad \frac{dy}{dx} = \frac{1}{1+x^2}.$$

$$21. y = \cos^{-1} \frac{1-x^2}{1+x^2}. \quad \frac{dy}{dx} = \frac{2}{1+x^2}.$$

$$22. y = \tan^{-1} \sqrt{\frac{1-\cos x}{1+\cos x}}. \quad \frac{dy}{dx} = \frac{1}{2}.$$

$$23. y = \frac{x \sin^{-1} x}{\sqrt[4]{(1-x^2)}} + \log \sqrt[4]{(1-x^2)}. \quad \frac{dy}{dx} = \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{4}}}.$$

$$24. y = (x+a) \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt[4]{(ax)}. \quad \frac{dy}{dx} = \tan^{-1} \sqrt{\frac{x}{a}}.$$

## IX.

*Differentials of Functions of Two Variables.*

**64.** The formulas already deduced enable us to differentiate any function of two variables, expressed by elementary functional symbols; the application of these formulas is, how-

ever, sometimes facilitated by a general principle which will now be shown to be applicable to such functions.

The formulas mentioned above involve differential factors of the first degree only. It follows, therefore, that the differentials resulting from their application consist of terms each of which contains the first power of the differential of one of the variables. In other words, if

$$u = f(x, y),$$

$$du = \phi(x, y)dx + \psi(x, y)dy. \quad . \quad . \quad . \quad (1)$$

Now, if  $y$  were constant, we should have  $dy = 0$ , and the value of  $du$  would reduce to that of the first term in the right-hand member of (1); hence this term may be found by *differentiating*  $u$  *on the supposition that*  $y$  *is constant*, and in like manner the second term can be found by differentiating  $u$  on the supposition that  $x$  is constant. The sum of the results thus obtained is therefore the required value of  $du$ .

65. As an example, let

$$z = u^v.$$

Were  $v$  constant, we should have for the value of  $dz$ , by formula (f), Art. 37,

$$v u^{v-1} du;$$

and, were  $u$  constant, we should have, by formula (h), Art. 46,

$$\log u \cdot u^v dv;$$

whence, adding these results,

$$dz = u^{v-1}(v du + u \log u dv).$$

Although this result has been obtained on the supposition that  $u$  and  $v$  are independent variables, it is evident that any two functions of a single variable may be substituted for  $u$  and  $v$ . Thus, if

$$u = nx \quad \text{and} \quad v = x^2,$$

we have

$$s = (nx)^{x^2},$$

and, on substituting,

$$\begin{aligned} ds &= (nx)^{x^2-1} (x^2 n dx + nx \log(nx) \cdot 2x dx), \\ &= x (nx)^{x^2} [1 + 2 \log(nx)] dx, \end{aligned}$$

which is identical with the expression obtained in Art. 47, for the differential of this function.

### Examples IX.

$$1. \ u = xy e^{x+2y}. \quad du = e^{x+2y} [y(1+x) dx + x(1+2y) dy].$$

$$2. \ u = \log \tan \frac{x}{y}. \quad du = 2 \frac{y dx - x dy}{y^2 \sin 2 \frac{x}{y}}.$$

$$3. \ u = \log \tan^{-1} \frac{x}{y}. \quad du = \frac{y dx - x dy}{(x^2 + y^2) \tan^{-1} \frac{x}{y}}.$$

$$\begin{aligned} 4. \ u &= \frac{\sqrt[4]{x} + \sqrt[4]{y}}{x + y}. \\ du &= \frac{[y - x - 2\sqrt[4]{xy}][\sqrt[4]{y} dx + [x - y - 2\sqrt[4]{xy}]\sqrt[4]{x} dy]}{2\sqrt[4]{xy}(x+y)^2}. \end{aligned}$$

$$5. \ u = \frac{e^x y}{(x^2 + y^2)^{\frac{1}{2}}}. \quad du = \frac{y e^x dx}{(x^2 + y^2)^{\frac{1}{2}}} + \frac{x(x dy - y dx) e^x}{(x^2 + y^2)^{\frac{3}{2}}}.$$

$$6. \ u = \tan^{-1} \frac{x-y}{x+y}. \quad du = \frac{y dx - x dy}{x^2 + y^2}.$$



$$7. u = \sqrt{\frac{x^2 - y^2}{x^2 + y^2}}. \quad du = \frac{2xy(y dx - x dy)}{(x^2 + y^2)^{\frac{3}{2}} \sqrt{x^2 - y^2}}.$$

$$8. u = \log \frac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}}. \quad du = \frac{2(y dx - x dy)}{y \sqrt{x^2 - y^2}}.$$

9. Given  $x = r \cos \theta$ , and  $y = r \sin \theta$ ; eliminate  $\theta$  and find  $dr$ ; also eliminate  $r$  and find  $d\theta$ .

$$dr = \frac{x dx + y dy}{\sqrt{x^2 + y^2}}, \text{ and } d\theta = \frac{x dy - y dx}{x^2 + y^2}.$$

### Miscellaneous Examples.

$$1. y = \frac{x}{\sqrt{1+x}}. \quad \frac{dy}{dx} = \frac{x+2}{2(1+x)^{\frac{3}{2}}}.$$

$$2. y = \sqrt{\frac{a^2 - x^2}{b^2 - x^2}}. \quad \frac{dy}{dx} = \frac{(a^2 - b^2)x}{(a^2 - x^2)^{\frac{3}{2}}(b^2 - x^2)^{\frac{3}{2}}}.$$

$$3. y = \frac{\sqrt{a+x}}{\sqrt{a} + \sqrt{x}}. \quad \frac{dy}{dx} = \frac{\sqrt{a}(\sqrt{x} - \sqrt{a})}{2\sqrt{x}\sqrt{a+x}(\sqrt{a} + \sqrt{x})^2}.$$

$$4. y = (\sqrt{x} - 2\sqrt{a})\sqrt{(\sqrt{a} + \sqrt{x})}. \quad \frac{dy}{dx} = \frac{3}{4\sqrt{(\sqrt{a} + \sqrt{x})}}.$$

$$5. y = \frac{(x-1)(e^x+1)e^x}{e^x-1}. \quad \frac{dy}{dx} = \frac{e^x(xe^{2x}-2xe^x+2e^x-x)}{(e^x-1)^2}.$$

$$6. y = \log \frac{(1+x^2)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} + \frac{1}{2} \tan^{-1}x. \quad \frac{dy}{dx} = \frac{x}{(1+x)(1+x^2)}.$$

$$7. y = \log \left( \frac{1+x}{1-x} \right)^{\frac{1}{2}} - \frac{1}{2} \tan^{-1}x. \quad \frac{dy}{dx} = \frac{x^2}{1-x^4}.$$

$$8. y = \log [x + \sqrt{(x^2 - a^2)}] + \sec^{-1} \frac{x}{a}. \quad \frac{dy}{dx} = \frac{1}{x} \sqrt{\frac{x+a}{x-a}}.$$

$$9. y = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} + \log \frac{1-x}{1+x}. \quad \frac{dy}{dx} = \frac{2x \sin^{-1} x}{(1-x^2)^{\frac{3}{2}}}.$$

$$10. y = \frac{(1 + 3x + 3x^2)^{\frac{1}{2}}}{x}, \quad \frac{dy}{dx} = -\frac{(1+x)^2}{x^3(1+3x+3x^2)^{\frac{3}{2}}}.$$

$$11. y = a \log \frac{a + \sqrt{a^2 - x^2}}{x} - \sqrt{a^2 - x^2}.$$

$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}.$$

$$12. y = \frac{(1 - x^2)^{\frac{3}{2}} \sin^{-1} x}{x}.$$

$$\frac{dy}{dx} = \frac{1 - x^2}{x} - \frac{1 + 2x^2}{x^3} \sqrt{1 - x^2} \sin^{-1} x.$$

$$13. y = \log \sqrt{\frac{1 - \cos x}{1 + \cos x}}, \quad \frac{dy}{dx} = \frac{1}{\sin x}.$$

$$14. y = \tan^{-1} \left[ \sqrt{\frac{a-b}{a+b}} \cdot \tan \frac{x}{2} \right], \quad \frac{dy}{dx} = \frac{\sqrt{a^2 - b^2}}{2(a + b \cos x)}.$$

$$15. y = \sec^{-1} \frac{1}{2x^2 - 1}, \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1 - x^2}}.$$

$$16. y = \cos^{-1} \frac{x^{2n} - 1}{x^{2n} + 1}, \quad \frac{dy}{dx} = -\frac{2nx^{n-1}}{x^{2n} + 1}.$$

$$17. y = a \cos^{-1} \frac{a-x}{b} - \sqrt{b^2 - (a-x)^2}.$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{b^2 - (a-x)^2}}.$$

$$18. y = \cos^{-1} x - 2\sqrt{\frac{1-x}{1+x}}, \quad \frac{dy}{dx} = \frac{\sqrt{1-x}}{(1+x)^{\frac{3}{2}}}.$$

$$19. y = \frac{ax - 1}{\sqrt{1+x^2}} e^{a \tan^{-1} x}, \quad \frac{dy}{dx} = \frac{(1+a^2)x}{(1+x^2)^{\frac{3}{2}}} e^{a \tan^{-1} x}.$$

Use logarithmic differentials.

## CHAPTER IV.

### SUCCESSIVE DIFFERENTIATION.

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#### X.

#### *Velocity and Acceleration.*

66. If the variable quantity  $x$  represent the distance of a point, moving in a straight line, from a fixed origin taken on the line, the rate of  $x$  will represent the velocity of the point.

Denoting this velocity by  $v_x$  we have, in accordance with the definition given in Art. 17,

$$v_x = \frac{dx}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this expression the arbitrary interval of time  $dt$  is regarded as constant, while  $dx$ , and consequently  $v_x$ , is in general variable. Differentiating equation (1) we have, since  $dt$  is constant,

$$dv_x = \frac{d(dx)}{dt}.$$

The differential of  $dx$ , denoted above by  $d(dx)$ , is called the *second differential* of  $x$ ; it is usually written in the abbreviated form  $d^2x$ , and read "d-second  $x$ ." The rate of  $v_x$  is therefore expressed thus:—

$$\frac{dv_x}{dt} = \frac{d^2x}{(dt)^2}.$$

The rate of the velocity of a point is called its *acceleration*, and is usually denoted by  $\alpha$ ; hence we write

$$\alpha_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad \dots \dots \dots (2)$$

the marks of parenthesis being usually omitted in the denominator of this expression.

**67.** When the space  $x$  described by a moving point is a given function of the time  $t$ , the derivative of this function is, by equation (1), an expression for the velocity in terms of  $t$ . The derivative of the latter expression, which is called the *second derivative* of  $x$ , is therefore, by equation (2), an expression for the acceleration in terms of  $t$ .

A *positive* value of the acceleration  $\alpha$  indicates an *algebraic increase* of the velocity  $v$ , whether the latter be positive or negative; and, on the other hand, a negative value of  $\alpha$  indicates an *algebraic decrease* of the velocity.

**68.** As an illustration, let  $x$  denote the space which a body falling freely describes in the time  $t$ . A well-known mechanical formula gives

$$x = \frac{1}{2}gt^2. \quad \dots \dots \dots (1)$$

Hence we derive 
$$v_x = \frac{dx}{dt} = gt, \quad \dots \dots \dots (2)$$

and 
$$\alpha_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = g. \quad \dots \dots \dots (3)$$

In this case, therefore, the acceleration is constant and positive, and accordingly  $v_x$ , which is likewise positive, is numerically increasing.

**69.** When the velocity is given in terms of  $x$ , the acceleration can readily be expressed in terms of the same variable, as in the following example.

Given  $v_x = 2 \sin x$ ;

whence  $\frac{dv_x}{dt} = 2 \cos x \frac{dx}{dt}$ ;

that is,  $\alpha_x = 2 \cos x \cdot v_x = 4 \cos x \sin x = 2 \sin 2x$ .

The general expression for  $\alpha_x$ , when  $v_x$  is given in terms of  $x$ , is

$$\alpha_x = \frac{dv_x}{dt} = \frac{dv_x}{dx} \frac{dx}{dt} = v_x \frac{dv_x}{dx} = \frac{1}{2} \frac{d(v_x^2)}{dx} \dots (1)$$

### *Component Velocities and Accelerations.*

70. When the motion of a point is not rectilinear but is nevertheless confined to a plane, its position is referred to co-ordinate axes; the coordinates,  $x$  and  $y$ , are evidently functions of  $t$ , and the derivatives  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ , which denote the rates of these variables, are called the *component* or *resolved velocities* in the directions of the axes. Denoting these component velocities by  $v_x$  and  $v_y$ , we have

$$v_x = \frac{dx}{dt}, \text{ and } v_y = \frac{dy}{dt}.$$

Again, denoting by  $s$  the actual space described, as measured from some fixed point of the path,  $s$  will likewise be a function of  $t$ , and the derivative  $\frac{ds}{dt}$  will denote the actual velocity of the point. (Compare Art. 48.) Now, the axes being rectangular, and  $\phi$  denoting the inclination of the direction of the motion to the axis of  $x$ , we have

$$dx = ds \cos \phi, \text{ and } dy = ds \sin \phi.$$

Hence,  $\frac{dx}{dt} = \frac{ds}{dt} \cos \phi$ , and  $\frac{dy}{dt} = \frac{ds}{dt} \sin \phi$ ;

or  $v_x = v \cos \phi$ , and  $v_y = v \sin \phi$ .

Squaring and adding,

$$v_x^2 + v_y^2 = v^2.$$

The last equation enables us to determine from the component velocities the actual velocity in the curve.

71. If we represent the accelerations of the resolved motions in the directions of the axes by  $\alpha_x$  and  $\alpha_y$ , we shall have, by Art. 66,

$$\alpha_x = \frac{d^2x}{dt^2} \quad \text{and} \quad \alpha_y = \frac{d^2y}{dt^2}.$$

These accelerations,  $\alpha_x$  and  $\alpha_y$ , will be positive when the resolved motions are *accelerated in the positive directions of the corresponding axes*; that is, when they increase a positive resolved velocity, or numerically decrease a negative resolved velocity.

### Examples X.

1. The space in feet described in the time  $t$  by a point moving in a straight line is expressed by the formula

$$x = 48t - 16t^2;$$

find the acceleration, and the velocity at the end of  $2\frac{1}{2}$  seconds; also find the value of  $t$  for which  $v = 0$ .

$$\alpha = -32; \quad v = 0, \quad \text{when } t = 1\frac{1}{2}.$$

2. If the space described in  $t$  seconds be expressed by the formula

$$x = 10 \log \frac{4}{4+t};$$

find the velocity and acceleration at the end of 1 second, and at the end of 16 seconds.

$$\text{When } t = 1, \quad v = -2 \quad \text{and} \quad \alpha = \frac{2}{3}.$$

3. If a point moves in a fixed path so that

$$s = \sqrt{t},$$

show that the acceleration is negative and proportional to the cube of the velocity. Find the value of the acceleration at the end of one second, and at the end of nine seconds.  $-\frac{1}{8}$ , and  $-\frac{1}{108}$ .

4. If a point move in a straight line so that

$$x = a \cos \frac{1}{2}\pi t,$$

show that

$$\alpha = -\frac{1}{4}\pi^2 x.$$

5. If

$$x = a e^t + b e^{-t},$$

prove that

$$\alpha = x.$$

6. If a point referred to rectangular coordinate axes move so that

$$x = a \cos t + b \quad \text{and} \quad y = a \sin t + c,$$

show that its velocity will be uniform. Find the equation of the path described.

*Eliminate  $t$  from the given equations.*

7. A projectile moves in the parabola whose equation is

$$y = x \tan \alpha - \frac{g}{2 V^2 \cos^2 \alpha} x^2,$$

(the axis of  $y$  being vertical) with a uniform horizontal velocity

$$v_x = V \cos \alpha;$$

find the velocity in the curve, and the vertical acceleration.

$$v = \sqrt{V^2 - 2gy}, \text{ and } \alpha_y = -g.$$

8. A point moves in the curve, whose equation is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

so that  $v_x$  is constant and equal to  $k$ ; find the acceleration in the direction of the axis of  $y$ .

$$\alpha_y = \frac{a^{\frac{1}{2}} k^2}{3x^{\frac{1}{2}} y^{\frac{1}{2}}}.$$

9. If a point move so that  $v = \sqrt{2gx}$ ; determine the acceleration.

Use equation (1), Art. 69.

$$\alpha = g.$$

10. If a point move so that we have

$$v^2 = c - \mu \log x,$$

determine the acceleration.

$$\alpha = -\frac{\mu}{2x}.$$

11. If a point move so that we have

$$v^2 = c + \frac{2\mu}{\sqrt{(x^2 + b^2)}},$$

determine the acceleration.

$$\alpha = -\frac{\mu x}{(x^2 + b^2)^{\frac{3}{2}}}.$$

12. The velocity of a point is inversely proportional to the square of its distance from a fixed point of the straight line in which it moves, the velocity being 2 feet per second when the distance is six inches; determine the acceleration at a given distance  $s$  from the fixed point.

$$-\frac{1}{2s^3} \text{ feet.}$$

13. The velocity of a point moving in a straight line is  $m$  times its distance from a fixed point at the perpendicular distance  $a$  from the straight line; determine the acceleration at the distance  $x$  from the foot of the perpendicular.

$$\alpha = m^2 x.$$

14. The relation between  $x$  and  $t$  being expressed by

$$t \sqrt{\frac{2\mu}{a}} = \sqrt{ax - x^2} - \frac{1}{2} a \operatorname{vers}^{-1} \frac{2x}{a};$$

find the acceleration in terms of  $x$ .

$$\alpha = -\frac{\mu}{x^3}.$$

15. A point moves in the hyperbola

$$y^2 = p^2 x^2 + q^2$$

in such a manner that  $v_x$  has the constant value  $c$ ; prove that



$$v_y^2 = p^2 c^2 - \frac{p^2 c^2 q^2}{y^2},$$

and thence derive  $\alpha_y$  by equation (1), Art. 69.

$$\alpha_y = \frac{p^2 c^2 q^2}{y^3}.$$

16. A point describes the conic section

$$y^2 = 2mx + nx^3,$$

$v_x$  having the constant value  $c$ ; determine the value of  $\alpha_y$ .

*Express  $v_y^2$  in terms of  $y$ , and proceed as in Example 15.*

$$\alpha_y = -\frac{m^2 c^2}{y^3}.$$

## XI.

### *Successive Derivatives.*

**72.** The derivative of  $f(x)$  is another function of  $x$ , which we have denoted by  $f'(x)$ ; if we take the derivative of the latter, we obtain still another function of  $x$ , which is called the second derivative of the original function  $f(x)$ , and is denoted by  $f''(x)$ . Thus if

$$f(x) = x^3, \quad f'(x) = 3x^2, \quad \text{and} \quad f''(x) = 6x.$$

Similarly the derivative of  $f''(x)$  is denoted by  $f'''(x)$ , and is called the third derivative of  $f(x)$ ; etc. When one of these successive derivatives has a constant value, the next and all succeeding derivatives evidently vanish. Thus, in the above example,  $f'''(x) = 6$ , consequently, in this case,  $f^{(4)}(x)$  and all higher derivatives vanish.

### *The Geometrical Meaning of the Second Derivative.*

**73.** If the curve whose equation is

$$y = f(x)$$

be constructed, we have seen (Art. 26) that

$$\frac{dy}{dx} = f'(x) = \tan \phi,$$

$\phi$  being the inclination of the curve to the axis of  $x$ ; hence

$$f''(x) = \frac{d(\tan \phi)}{dx}.$$

If now the value of this derivative be *positive*,  $\tan \phi$  will be an *increasing* function of  $x$ , as in Fig. 6, in which, as we proceed toward the right,  $\tan \phi$  (at first negative) increases algebraically throughout. In this case, therefore, the curve appears *concave when viewed from above*. On the other hand, if  $f''(x)$  be *negative*,  $\tan \phi$  will be a decreasing function of  $x$ , as in Fig. 7, in which, as we proceed toward the right,  $\tan \phi$  decreases algebraically throughout, the curve appearing *convex when viewed from above*.



FIG. 6.



FIG. 7.

**74.** A point which separates a concave from a convex portion of a curve is called a *point of inflexion*, or a *point of contrary flexure*.

It is obvious from the preceding article that, at a point of inflexion, like  $P$  in Fig. 8,  $f''(x)$  must *change sign*; hence at such a point, the value of this derivative must become either zero or infinity.

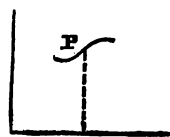


FIG. 8.

**75.** When a curve is described by a moving point, the character of the curvature is dependent upon the component accelerations of the motion. For, if we put

$$v_x = c, \quad \text{or} \quad dx = c dt,$$

$c$  denoting a constant, we have

$$f''(x) = \frac{d^2y}{c dt^2};$$

and hence 
$$f''(x) = \frac{1}{c^2} \cdot \frac{d^2y}{dt^2} = \frac{\alpha_y}{c^2}.$$

Whence it follows that, if  $v_x$  is constant,  $\alpha_y$  and  $f''(x)$  have the same sign, and consequently that a portion of a curve which is concave when viewed from above is one in which  $\alpha_y$  is positive when  $\alpha_x$  is zero.

### *Successive Differentials.*

76. The successive differentials of a function of  $x$  involve the successive differentials of  $x$ ; thus, if

$$y = x^3,$$

we have  $dy = 3x^2 dx,$

$$d^2y = 6x(dx)^2 + 3x^2 d^2x,$$

and  $d^3y = 6(dx)^3 + 18x dx d^2x + 3x^2 d^3x.$

In general, if

$$y = f(x),$$

$$dy = f'(x) dx,$$

$$d^2y = f''(x) (dx)^2 + f'(x) d^2x,$$

and  $d^3y = f'''(x) (dx)^3 + 3f''(x) dx d^2x + f'(x) d^3x.$

### *Equicrescent Variables.*

77. A variable is said to be *equicrescent* when its rate is constant; since  $dt$  in the expression  $\frac{dx}{dt}$  is assumed to be constant,  $dx$  is also constant, when  $x$  is equicrescent.

In expressing the differentials of a function, it is admissible

to assume the independent variable to be equicrescent, since the differential of this variable is arbitrary. This hypothesis greatly simplifies the expressions for the second and higher differentials of functions of  $x$ , inasmuch as it is evidently equivalent to making all differentials of  $x$  higher than the first vanish. Thus, in the general expressions for  $d^2y$  and  $d^3y$  given in the preceding article, all the terms except the first disappear, and it is easy to see that, in general, we shall have

$$d^n y = f^n(x) (dx)^n,$$

when  $x$  is equicrescent.

78. From the above equation we derive

$$\frac{d^n y}{dx^n} = f^n(x).$$

The expression in the first member of this equation is the usual symbol for the  $n$ th derivative of  $y$  regarded as a function of  $x$ . The  $n$ th *differential* which occurs in this symbol is always understood to denote the value which this differential assumes *when the variable indicated in the denominator is equicrescent*.

The symbol  $\frac{d}{dx}$  is frequently used to denote the operation of taking the derivative with reference to  $x$ , and similarly the symbol  $\left(\frac{d}{dx}\right)^n$ , or  $\frac{d^n}{dx^n}$ , is used to denote the operation of taking the derivative with respect to  $x$ ,  $n$  times in succession.

### Examples XI.

1. Find the second derivative of  $\sec x$ , and distinguish the concave from the convex portions of the curve  $y = \sec x$ . Also show that the curve  $y = \log x$  is everywhere convex.

2. Find the points of inflexion in the curve  $y = \sin x$ .

3. Find the point of inflexion of the curve

$$y = 2x^3 - 3x^2 - 12x + 6.$$

The point is  $(\frac{1}{2}, -\frac{1}{2})$ .

4. Show that the curve  $y = \tan x$  is concave when  $y$  is positive, and convex when  $y$  is negative.

5. Find the points of inflexion of the curve

$$y = x^4 - 2x^3 - 12x^2 + 11x + 24.$$

The points are  $(2, -2)$  and  $(-1, 4)$ .

6. If  $f(x) = \frac{1+x}{1-x}$ , find  $f^n(x)$ .  $f^n(x) = \frac{240}{(1-x)^5}$ .

7. If  $f(x) = \frac{a}{x^n}$ , find  $f'''(x)$ .  $f'''(x) = -\frac{n(n+1)(n+2)a}{x^{n+3}}$ .

8. If  $y$  is a function of  $x$  of the form

$$Ax^n + Bx^{n-1} + \dots + Mx + N,$$

prove that  $\frac{d^n y}{dx^n} = 1 \cdot 2 \cdot 3 \dots n A$ .

9. If  $f(x) = b^{ax}$ , find  $f^n(x)$ .  $f^n(x) = a^n (\log b)^n b^{ax}$ .

10. If  $f(x) = x^3 \log(mx)$ , find  $f^{iv}(x)$ .  $f^{iv}(x) = \frac{6}{x}$ .

11. If  $f(x) = \log \sin x$ , find  $f'''(x)$ .  $f'''(x) = \frac{2 \cos x}{\sin^3 x}$ .

12. If  $f(x) = \sec x$ , find  $f''(x)$  and  $f'''(x)$ .

$$f''(x) = 2 \sec^3 x - \sec x, \text{ and } f'''(x) = \sec x \tan x (6 \sec^2 x - 1).$$

13. If  $f(x) = \tan x$ , find  $f'''(x)$  and  $f^{iv}(x)$ .

$$f'''(x) = 6 \sec^4 x - 4 \sec^2 x, \text{ and } f^{iv}(x) = 8 \tan x \sec^2 x (3 \sec^2 x - 1).$$

14. If  $f(x) = x^x$ , find  $f''(x)$ .  $f''(x) = x^x (1 + \log x)^2 + x^{x-1}$ .

15. If  $y = \varepsilon^{\frac{1}{x}}$ , find  $\frac{d^2y}{dx^2}$ .  $\frac{d^2y}{dx^2} = -\frac{1}{x^2} (1 + 6x + 6x^2) \varepsilon^{\frac{1}{x}}$ .

16. If  $y = \varepsilon^{-x^2}$ , find  $\frac{d^2y}{dx^2}$ .  $\frac{d^2y}{dx^2} = 4x(3 - 2x^2) \varepsilon^{-x^2}$ .

17. If  $y = \log(\varepsilon^x + \varepsilon^{-x})$ , find  $\frac{d^2y}{dx^2}$ .  $\frac{d^2y}{dx^2} = -8 \frac{\varepsilon^x - \varepsilon^{-x}}{(\varepsilon^x + \varepsilon^{-x})^2}$ .

18. If  $y = \frac{1}{\varepsilon^x - 1}$ , find  $\frac{d^2y}{dx^2}$  and  $\frac{d^4y}{dx^4}$ .

$$\frac{d^2y}{dx^2} = \frac{\varepsilon^{2x} + \varepsilon^x}{(\varepsilon^x - 1)^3}, \text{ and } \frac{d^4y}{dx^4} = \frac{\varepsilon^x + 11\varepsilon^{2x} + 11\varepsilon^{3x} + \varepsilon^{4x}}{(\varepsilon^x - 1)^5}.$$

19. If  $y = \sin^{-1} x$ , find  $\frac{d^4y}{dx^4}$ .  $\frac{d^4y}{dx^4} = \frac{9x + 6x^3}{(1 - x^2)^{\frac{5}{2}}}$ .

20. If  $y = \varepsilon^{\sin x}$ , find  $\frac{d^3y}{dx^3}$ .

$$\frac{d^3y}{dx^3} = -\varepsilon^{\sin x} \cos x \sin x (\sin x + 3).$$

21. If  $y = \frac{x}{1 + \log x}$ , find  $\frac{d^2y}{dx^2}$ .  $\frac{d^2y}{dx^2} = \frac{1 - \log x}{x(1 + \log x)^3}$ .

22. Find the value of  $d^2(\varepsilon^x)$ , when  $x$  is not equicrescent.

$$d^2(\varepsilon^x) = \varepsilon^x(dx)^2 + 3\varepsilon^x d^2x dx + \varepsilon^x d^2x.$$

23. Find the value of  $\frac{d^3}{dt^3}(\sin \theta)$ ,  $\theta$  being a function of  $t$ .

$$\frac{d^3}{dt^3}(\sin \theta) = -\cos \theta \left(\frac{d\theta}{dt}\right)^3 - 3 \sin \theta \frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} + \cos \theta \frac{d^3\theta}{dt^3}.$$

## CHAPTER V.

### THE EVALUATION OF INDETERMINATE FORMS.

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#### XII.

#### *Indeterminate or Illusory Forms.*

**79.** WHEN a function is expressed in the form of a fraction each of whose terms is variable, it may happen that, for a certain value of the independent variable, both terms reduce to zero. The function then takes the form  $\frac{0}{0}$ , and is said to be *indeterminate*, since its value cannot be ascertained by the ordinary process of dividing the value of the numerator by that of the denominator. The function has, nevertheless, a value as determinate for this as for any other value of the independent variable. It is the object of this chapter to show that such definite values exist, and to explain the methods by which they are determined.

The term *illusory form* is often used as synonymous with *indeterminate form*, and these terms are applied indifferently, not only to the form  $\frac{0}{0}$ , but also to the forms  $\frac{\infty}{\infty}$ ,  $\infty \cdot 0$ ,  $\infty - \infty$ , and to certain others whose logarithms assume the form  $\infty \cdot 0$ .

When a function of  $x$  takes an illusory form for  $x=a$ , the corresponding value of the function is sometimes called its *limiting value* as  $x$  approaches the value  $a$ .

**80.** The values of functions which assume illusory forms may

sometimes be ascertained by making use of certain algebraic transformations. Thus, for example, the function

$$\frac{a - \sqrt{a^2 - bx}}{x}$$

takes the form  $\frac{0}{0}$  when  $x = 0$ .

Multiplying both terms by the complementary surd

$$a + \sqrt{a^2 - bx},$$

we obtain 
$$\frac{bx}{x[a + \sqrt{a^2 - bx}]} = \frac{b}{a + \sqrt{a^2 - bx}}.$$

The last form is not illusory for the given value of  $x$ , since the factor which becomes zero has been removed from both terms of the fraction. The value of the fraction for  $x = 0$  is evidently  $\frac{b}{2a}$ .

The following notation is used to indicate this and similar results; viz.,

$$\left. \frac{a - \sqrt{a^2 - bx}}{x} \right]_0 = \frac{b}{2a},$$

the subscript denoting that value of the independent variable for which the function is evaluated.

### *Evaluation by Differentiation.*

81. Let  $\frac{v}{u}$  represent a function in which both  $u$  and  $v$  are functions of  $x$ , which vanish when  $x = a$ ; in other words, for this value of  $x$ , we have  $u = 0$ , and  $v = 0$ .



Let  $P$  be a moving point of which the abscissa and ordinate are simultaneous values of  $u$  and  $v$  ( $x$  not being represented in the figure); then, denoting the angle  $POU$  by  $\theta$ , and the inclination of the motion of  $P$  to the axis of  $u$  by  $\phi$ , we have

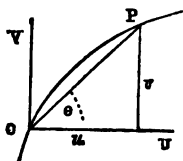


FIG. 9.

$$\tan \theta = \frac{v}{u}, \quad \text{and} \quad \tan \phi = \frac{dv}{du}.$$

At the instant when  $x$  passes through the value  $a$ ,  $u$  and  $v$  being zero by the hypothesis,  $P$  passes through the origin; the corresponding value of  $\theta$  is evidently determined by the direction in which  $P$  is moving at that instant, and is therefore equal to the value of  $\phi$  at that point.

Hence the values of  $\tan \theta$  and  $\tan \phi$  corresponding to  $x = a$  are equal, or

$$\left[ \frac{v}{u} \right]_{x=a} = \left[ \frac{dv}{du} \right]_{x=a};$$

therefore, to determine the value of  $\frac{v}{u}$  for  $x = a$ , we substitute for it the function  $\frac{dv}{du}$ , whose value is the same as that of the given function, when  $x = a$ .

82. This result may also be expressed in the following manner: let  $f(x)$  and  $\phi(x)$  be two functions, such that  $f(a) = 0$ , and  $\phi(a) = 0$ ; then

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)} \cdot \dots \dots \dots (1)$$

As an illustration, let us take  $\frac{\log x}{x-1}$ . When  $x = 1$ , this function takes the form  $\frac{0}{0}$ ; by the above process, we have

$$\left. \frac{\log x}{x-1} \right]_1 = \left. \frac{x^{-1}}{1} \right]_1 = 1,$$

the required value.

**83.** Since the substituted function  $\frac{dv}{du}$  or  $\frac{f'(x)}{\phi'(x)}$  frequently takes the indeterminate form, several repetitions of the process are sometimes requisite before the value of the function can be ascertained.

For example, the function  $\frac{1 - \cos \theta}{\theta^2}$  takes the form  $\frac{0}{0}$  when  $\theta = 0$ ; employing the process for evaluating, we have

$$\left. \frac{1 - \cos \theta}{\theta^2} \right]_0 = \left. \frac{\sin \theta}{2\theta} \right]_0,$$

which is likewise indeterminate; but, by repeating the process, we obtain

$$\left. \frac{1 - \cos \theta}{\theta^2} \right]_0 = \left. \frac{\sin \theta}{2\theta} \right]_0 = \left. \frac{\cos \theta}{2} \right]_0 = \frac{1}{2}.$$

**84.** If the given function, or any of the substituted functions, contains a factor which does not take the indeterminate form, this factor may be evaluated at once, as in the following example.

The function

$$\frac{(1-x)e^x - 1}{\tan^2 x}$$

is indeterminate for  $x = 0$ . By employing the usual process once, we obtain

$$\left. \frac{(1-x)e^x - 1}{\tan^2 x} \right]_0 = \left. \frac{-xe^x}{2 \sec^2 x \tan x} \right]_0,$$

which is likewise indeterminate; but, before repeating the process, we may evaluate the factor  $-\frac{e^x}{2 \sec^2 x}$ . The value of this factor is  $-\frac{1}{2}$ ; hence we write

$$\begin{aligned} \left[ \frac{(1-x)e^x - 1}{\tan^2 x} \right]_0 &= - \left[ \frac{x e^x}{2 \sec^2 x \tan x} \right]_0 = - \frac{1}{2} \frac{x}{\tan x} \\ &= - \frac{1}{2} \left[ \frac{1}{\sec^2 x} \right]_0 = - \frac{1}{2}. \end{aligned}$$

**85.** When the given function can be decomposed into factors each of which takes the indeterminate form, these factors may be evaluated separately. Thus, if the given function be

$$\frac{(e^x - 1) \tan^2 x}{x^3},$$

the form

$$\left( \frac{\tan x}{x} \right)^2 \left( \frac{e^x - 1}{x} \right)$$

may be employed. We have

$$\left[ \frac{\tan x}{x} \right]_0 = 1, \text{ and } \left[ \frac{e^x - 1}{x} \right]_0 = 1;$$

hence the value of the given function is unity.

When this method is used, if one of the factors is found to take the value zero while another is infinite, their product, being of the form  $0 \cdot \infty$ , must be treated by the usual method, since  $0 \cdot \infty$  is itself an illusory form.

**86.** Another mode of decomposing a given function is that of separating it into parts, and substituting the values of such parts as are found on evaluation to be finite.

As an illustration, we take the expression,

$$u_0 = \left[ \frac{(e^x - e^{-x})^2 - 2x^2(e^x + e^{-x})}{x^4} \right]_0.$$

Each of the fractions into which this function can be decomposed being obviously infinite, we first apply the usual process, thus obtaining

$$u_0 = \frac{2(\varepsilon^x - \varepsilon^{-x})(\varepsilon^x + \varepsilon^{-x}) - 4x(\varepsilon^x + \varepsilon^{-x}) - 2x^2(\varepsilon^x - \varepsilon^{-x})}{4x^3} \Big|_0.$$

Separating this expression into two fractions, thus,—

$$u_0 = \frac{(\varepsilon^x + \varepsilon^{-x})(\varepsilon^x - \varepsilon^{-x} - 2x)}{2x^3} \Big|_0 - \frac{\varepsilon^x - \varepsilon^{-x}}{2x} \Big|_0;$$

the latter is found on evaluation to have a finite value, and the expression reduces to

$$u_0 = \frac{\varepsilon^x - \varepsilon^{-x} - 2x}{x^3} \Big|_0 - 1.$$

Hence

$$u_0 = \frac{\varepsilon^x + \varepsilon^{-x} - 2}{3x^3} \Big|_0 - 1 = \frac{\varepsilon^x - \varepsilon^{-x}}{6x} \Big|_0 - 1 = -\frac{2}{3}.$$

### Examples XII.

1. Prove  $\frac{\sin x}{x} \Big|_0 = 1$ ,  $\frac{\tan x}{x} \Big|_0 = 1$ , and  $\frac{\varepsilon^x - 1}{x} \Big|_0 = 1$ .

*These results are frequently useful in evaluating other functions.*

Evaluate the following functions :

2.  $\frac{\varepsilon^x - \varepsilon^{-x}}{\log(1+x)},$  when  $x = 0.$  2.

3.  $\frac{a^n - x^n}{\log a - \log x},$   $x = a.$   $na^n.$

4.  $\frac{x^3 - 5x^2 + 7x - 3}{x^3 - x^2 - 5x - 3},$   $x = 3.$   $\frac{1}{4}.$

5.  $\frac{x^4 - 8x^3 + 22x^2 - 24x + 9}{x^4 - 4x^3 - 2x^2 + 12x + 9},$   $x = 3.$   $\frac{1}{4}.$

6.  $\frac{x\varepsilon^{2x} - \varepsilon^{2x} - x + 1}{\varepsilon^{2x} - 1},$   $x = 0.$   $-1.$

7.  $\frac{\sin x - \cos x}{\sin 2x - \cos 2x - 1}$ , when  $x = \frac{1}{2}\pi$ .  $\frac{1}{2}\sqrt{2}$ .
8.  $\frac{\log x}{\sqrt[4]{(1-x)}}$ ,  $x = 1$ . 0.
9.  $\frac{a^x - b^x}{x}$ ,  $x = 0$ .  $\log \frac{a}{b}$ .
10.  $\frac{\sqrt[4]{(1+x^2)(1-x)}}{1-x^2}$ , (See Art. 84),  $x = 1$ .  $\frac{\sqrt[4]{2}}{n}$ .
11.  $\frac{a^2 - x^2}{x^2} \left(1 - \cos \frac{x}{a}\right)$ ,  $x = 0$ .  $\frac{1}{2}$ .
12.  $\frac{\epsilon^{mx} - \epsilon^{ma}}{x - a}$ ,  $x = a$ .  $m\epsilon^{ma}$ .
13.  $\frac{a^{\sin x} - a}{\log \sin x}$ ,  $x = \frac{1}{2}\pi$ .  $a \log a$ .
14.  $\frac{1 - \cos x}{x \log(1+x)}$ ,  $x = 0$ .  $\frac{1}{2}$ .
15.  $\frac{\sqrt{x} \tan x}{(\epsilon^x - 1)^{\frac{1}{2}}}$ ,  $x = 0$ . 1.

Put in the form  $\sqrt{\frac{x}{\epsilon^x - 1}} \cdot \frac{\tan x}{x} \cdot \frac{x}{\epsilon^x - 1}$ . See Art. 85 and Example 1.

16.  $\frac{\sqrt{x} - \sqrt{a} + \sqrt{(x-a)}}{\sqrt{(x^2 - a^2)}}$ ,  $x = a$ .  $\frac{1}{\sqrt{(2a)}}$ .
17.  $\frac{x\sqrt{(3x - 2x^4)} - x^{\frac{5}{2}}}{1 - x^{\frac{3}{2}}}$ ,  $x = 1$ .  $\frac{81}{20}$ .
18.  $\frac{(a^2 + ax + x^2)^{\frac{1}{2}} - (a^2 - ax + x^2)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}} - (a-x)^{\frac{1}{2}}}$ ,  $x = 0$ .  $\sqrt[4]{a}$ .

Multiply both terms by the two complementary surds. See Art. 80.

$$19. \frac{(a^3 - x^3)^{\frac{1}{3}} + (a - x)^{\frac{2}{3}}}{(a^3 - x^3)^{\frac{1}{3}} + (a - x)^{\frac{1}{3}}}, \quad \text{when } x = a. \quad \frac{\sqrt[4]{2a}}{1 + a\sqrt[4]{3}}.$$

Divide both terms by  $(a - x)^{\frac{1}{3}}$ .

$$20. \frac{\sin x - x \cos x}{x - \sin x}, \quad x = 0. \quad 2.$$

$$21. \frac{e^x - e^{-x} - 2x}{x - \tan x}, \quad x = 0. \quad -1.$$

$$22. \frac{(x-2)e^x + x + 2}{x(e^x - 1)^2}, \quad x = 0. \quad \frac{1}{6}.$$

$$23. \frac{x^2 - x}{1 - x + \log x}, \quad x = 1. \quad -2.$$

$$24. \frac{\tan x - \sin x}{x^3}, \quad x = 0. \quad \frac{1}{2}.$$

Put in the form  $\left[ \frac{\sin x}{x} \right]_0 \cdot \left[ \frac{\sec x - 1}{x^2} \right]_0$ .

$$25. \frac{(x-1)^2 + \sin^2(x^2-1)^{\frac{1}{2}}}{(x+1)(x-1)^{\frac{1}{2}}}, \quad x = 1. \quad \sqrt[4]{2}.$$

$$26. \frac{1 - x + \log x}{1 - \sqrt[4]{2x - x^3}}, \quad x = 1. \quad -1.$$

$$27. \frac{\sin x - \log(e^x \cos x)}{x^2}, \quad x = 0. \quad \frac{1}{2}.$$

$$28. \frac{\frac{1}{2}\pi - \tan^{-1} x}{x^2 - e^{\sin(\log x)}}, \quad x = 1. \quad \frac{1}{2(1-\pi)}.$$

$$29. \frac{\tan(a+x) - \tan(a-x)}{\tan^{-1}(a+x) - \tan^{-1}(a-x)}, \quad x = 0. \quad (1+a^2) \sec^2 a.$$

$$30. \frac{x \sin x - \frac{1}{2}\pi}{\cos x}, \quad x = \frac{1}{2}\pi. \quad -1.$$

$$31. \frac{e^x - e^{\sin x}}{x - \sin x}, \quad \text{when } x = 0. \quad 1.$$

$$32. \frac{m^n \sin nx - n^m \sin mx}{\tan nx - \tan mx}, \quad m = n.$$

$$n^{n-1} (n \cos nx - \sin nx) \cos^2 nx.$$

*In solving this and the following example, x and n may be regarded as constants, and m as a variable.*

$$33. \frac{\tan nx - \tan mx}{\sin (n^2 x - m^2 x)}, \quad m = n. \quad \frac{\sec^2 nx}{2n}.$$

### XIII.

*The Form*  $\frac{\infty}{\infty}$ .

87. Let  $\frac{f(x)}{\phi(x)}$  denote a function which assumes the form  $\frac{\infty}{\infty}$  when  $x = a$ , then we have

$$\frac{f(x)}{\phi(x)} = \frac{\frac{1}{f(x)}}{\frac{1}{\phi(x)}}. \quad \dots \quad (1)$$

The second member of this equation takes the form  $\frac{0}{0}$  when  $x = a$ ; we therefore have, by equation (1) Art. 82,

$$\frac{f(a)}{\phi(a)} = \frac{\frac{1}{\phi(a)}}{\frac{1}{f(a)}} = \frac{-\frac{\phi'(a)}{[\phi(a)]^2}}{-\frac{f'(a)}{[f(a)]^2}} = \frac{\phi'(a)}{f'(a)} \left\{ \frac{f(a)}{\phi(a)} \right\}^2; \quad \dots \quad (2)$$

whence, if  $\frac{f(a)}{\phi(a)}$  is neither zero nor infinity, we infer that

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This formula, it will be observed, is identical with that employed when the function takes the form  $\frac{0}{0}$ .

**88.** When the value of  $\frac{f(a)}{\phi(a)}$  is either zero or infinity, equation (2), Art. 87, will be satisfied independently of the existence of equation (3); we are not justified therefore, when this is the case, in deriving the latter from the former. The following demonstration shows, however, that equation (3) holds in these cases also.

First, when the value of  $\frac{f(a)}{\phi(a)}$  is zero, by adding a finite quantity  $n$  to the given function, we have

$$\frac{f(a)}{\phi(a)} + n = \frac{f(a) + n\phi(a)}{\phi(a)},$$

a function which is by hypothesis finite. To this function therefore the demonstration given in Art. 87 applies; hence

$$\frac{f(a)}{\phi(a)} + n = \frac{f'(a) + n\phi'(a)}{\phi'(a)} = n + \frac{f'(a)}{\phi'(a)};$$

therefore  
as before.

$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)},$$

Again, if the value of  $\frac{f(a)}{\phi(a)}$  is infinite, that of  $\frac{\phi(a)}{f(a)}$  is zero, and, by the last result,

$$\frac{\phi(a)}{f(a)} = \frac{\phi'(a)}{f'(a)};$$

hence, in this case, likewise



$$\frac{f(a)}{\phi(a)} = \frac{f'(a)}{\phi'(a)}.$$

*Derivatives of Functions which assume an Infinite Value.*

**89.** When  $f(x)$  becomes infinite, for a finite value  $a$  of the independent variable,  $f'(a)$  is likewise infinite. For, let  $b$  denote a value of  $x$  so taken that  $f(x)$  shall be finite for  $x = b$  and for all values of  $x$  between  $b$  and  $a$ : then, as  $x$  varies from  $b$  to  $a$ , the rate of  $f(x)$  must assume an infinite value, otherwise  $f(x)$  would remain finite. The value of  $x$  for which the rate is infinite must be  $a$  or some value of  $x$  between  $b$  and  $a$ ; that is, some value of  $x$  nearer to  $a$  than  $b$  is. Now, since  $b$  may be taken as near as we please to  $a$ , the value of  $x$  for which the rate is infinite cannot differ from  $a$ . The expression for this rate is  $f'(x) \frac{dx}{dt}$ , in which  $\frac{dx}{dt}$  may be assumed finite, therefore  $f'(x)$  must be infinite when  $x = a$ ; in other words,  $f'(a)$  is infinite when  $f(a)$  is infinite.

**90.** It follows from the theorem proved in the preceding article that when  $a$  is finite the function obtained by the application of formula (3), Art. 87, takes the same form,  $\frac{\infty}{\infty}$ , as that assumed by the original function. Hence, except when the given value of  $x$  is infinite, the application of some other process, either to the original function or to one of the substituted functions, is always requisite. Thus in the example,

$$\left. \frac{\log(\sin 2x)}{\log \sin x} \right]_0 = \frac{\infty}{\infty};$$

by using the above formula we obtain

$$\left[ \frac{\log \sin 2x}{\log \sin x} \right]_0 = \left[ \frac{2 \cot 2x}{\cot x} \right]_0,$$

which takes the form  $\frac{\infty}{\infty}$ ; but the last expression is equivalent to  $2 \left[ \frac{\sin x \cos 2x}{\sin 2x \cos x} \right]_0$ , and is therefore easily shown to have the value unity.

### *The Form 0. ∞.*

91. A function which takes this form may, by introducing the reciprocal of one of the factors, be so transformed as to take either of the forms  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ , as may be found most convenient. For example, let us take the function

$$x^{-n} \varepsilon^x,$$

which assumes the above form when  $x = \infty$ ,  $n$  being positive. In this case it is necessary to reduce to the form  $\frac{\infty}{\infty}$ . Thus—

$$x^{-n} \varepsilon^x = \frac{\varepsilon^x}{x^n} \Bigg|_{\infty} = \frac{\varepsilon^x}{nx^{n-1}} \Bigg|_{\infty} = \frac{\varepsilon^x}{n(n-1)x^{n-2}} \Bigg|_{\infty}, \text{ etc.}$$

By continuing this process, we finally obtain a fraction whose denominator is finite while its numerator is still infinite. Hence we have, for all finite values of  $n$ ,

$$x^{-n} \varepsilon^x \Big|_{\infty} = \infty.$$

### *The Form ∞ - ∞.*

92. A function which assumes this form may be so transformed as to take the form  $\frac{0}{0}$ . Let the given function be

$$\left[ \frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2} \right]_0,$$

which takes the form  $\infty - \infty$ , since the second term is easily shown to be infinite. But

$$\begin{aligned} \left[ \frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2} \right]_0 &= \frac{x - (1+x) \log(1+x)}{x^2(1+x)} \Big|_0 \\ &= \frac{x - (1+x) \log(1+x)}{x^2} \Big|_0 \\ &= \frac{1 - \log(1+x) - 1}{2x} \Big|_0 = -\frac{1}{2}. \end{aligned}$$

### Examples XIII.

Evaluate the following functions :

$$1. \frac{\sec x}{\sec 3x}, \quad \text{when } x = \frac{1}{3}\pi. \quad -3.$$

$$2. \frac{a^x}{\operatorname{cosec}(ma^{-x})}, \quad x = \infty. \quad m.$$

$$3. \frac{\log x}{x^n} \quad (n > 0), \quad x = \infty. \quad 0.$$

$$4. \frac{\tan x}{\log(x - \frac{1}{2}\pi)}, \quad x = \frac{1}{2}\pi. \quad \infty.$$

$$5. \frac{\sec(\frac{1}{2}\pi x)}{\log(1-x)}, \quad x = 1. \quad \infty.$$

$$6. \frac{\log \cos(\frac{1}{2}\pi x)}{\log(1-x)}, \quad x = 1. \quad 1.$$

- |                                                                |                       |                   |
|----------------------------------------------------------------|-----------------------|-------------------|
| 7. $\frac{\tan x}{\tan 3x},$                                   | $x = \frac{1}{2}\pi.$ | 3.                |
| 8. $\frac{\log(1+x)}{x},$                                      | $x = \infty.$         | 0.                |
| 9. $\left(a^{\frac{1}{x}} - 1\right)x,$                        | $x = \infty.$         | $\log a.$         |
| 10. $\frac{x^3 - a^3}{a^3} \tan \frac{\pi x}{2a},$             | $x = a.$              | $-\frac{4}{\pi}.$ |
| 11. $x^m (\log x)^n,$ ( $m$ and $n$ being positive),           | $x = 0.$              | 0.                |
| 12. $e^x \sin \frac{1}{x},$                                    | $x = \infty.$         | $\infty.$         |
| 13. $e^{-\frac{1}{x}} (1 - \log x),$                           | $x = 0.$              | 0.                |
| 14. $\sec \frac{\pi x}{2} \cdot \log \frac{1}{x},$             | $x = 1.$              | $\frac{2}{\pi}.$  |
| 15. $\frac{\log \tan nx}{\log \tan x},$                        | $x = 0.$              | 1.                |
| 16. $\frac{\log \cot \frac{x}{2}}{\cot x + \log x},$           | $x = 0.$              | 0.                |
| 17. $\sec x (x \sin x - \frac{1}{2}\pi),$                      | $x = \frac{1}{2}\pi.$ | $-1.$             |
| 18. $\log \left(2 - \frac{x}{a}\right) \tan \frac{\pi x}{2a},$ | $x = a.$              | $\frac{2}{\pi}.$  |
| 19. $(1-x) \tan \left(\frac{1}{2}\pi x\right),$                | $x = 1.$              | $\frac{2}{\pi}.$  |
| 20. $\log(x-a) \tan(x-a),$                                     | $x = a.$              | 0.                |

## XIV.

*Functions whose Logarithms take the Form  $\infty \cdot 0$ .*

**93.** In the case of a function of the form  $u^v$ , we have

$$\log u^v = v \log u.$$

The expression  $v \log u$  takes the illusory form  $0 \cdot \infty$  in two cases: first, when  $v = 0$  and  $\log u = \infty$ ; and secondly, when  $v = \infty$  and  $\log u = 0$ .

$\log u$  is infinite when  $u = 0$ , and also when  $u = \infty$ ; therefore the first case will arise when the original function takes one of the forms  $\infty^0$  or  $0^0$ .

$\log u = 0$  when  $u = 1$ , therefore the second case will arise when the original function takes the form  $1^\infty$ .

Hence functions which take either of the three illusory forms.

$$\infty^0, \quad 0^0, \quad \text{or} \quad 1^\infty,$$

may be evaluated by first evaluating their logarithms, which take the form  $0 \cdot \infty$ .

It is to be noticed however that  $0^\infty$  and  $\infty^\infty$  are not illusory forms, since their logarithms take the form  $\infty (\mp \infty)$ .

*The Form  $1^\infty$ .*

**94.** As an illustration of this form, we take the function  $\left(1 + \frac{a}{x}\right)^x$ , which assumes the form  $1^\infty$  when  $x = \infty$ . Denoting this function by  $u$ , we have

$$\log u = x \log \left(1 + \frac{a}{x}\right) = \frac{\log \left(1 + \frac{a}{x}\right)}{\frac{1}{x}},$$

the last expression assuming the form  $\frac{0}{0}$  when  $x = \infty$ .

In evaluating this logarithm, it is convenient to substitute  $z$  for  $\frac{1}{x}$ ; then

$$\log u_{\infty} = \frac{\log(1+az)}{z} \Big]_0,$$

since, when  $x = \infty$ ,  $z = 0$ . Taking derivatives, we have

$$\log u_{\infty} = \frac{\log(1+az)}{z} \Big]_0 = \frac{a}{1+az} \Big]_0 = a.$$

Hence 
$$u_{\infty} = \left(1 + \frac{a}{x}\right)^x \Big]_{\infty} = e^a.$$

95. If  $a = 1$ , we have

$$\left(1 + \frac{1}{x}\right)^x \Big]_{\infty} = e;$$

that is, as  $x$  increases indefinitely, the *limiting value* of the function  $\left(1 + \frac{1}{x}\right)^x$  is  $e$ . The Napierian base is often defined as the limiting value of this function, or, what is the same thing, by formula

$$e = \left(1 + \frac{1}{x}\right)^x \Big]_{\infty}.$$

### *The Form 0°.*

96. The function  $x^x \Big]_0$ , by the aid of which many functions of similar form may be evaluated, will serve as an illustration of the form 0°.

Let

$$u = x^x;$$

then

$$\log u = x \log x,$$

and  $\log u \Big|_0 = \frac{\log x}{x^{-1}} \Big|_0 = -\frac{x^{-1}}{x^{-2}} \Big|_0 = 0;$

therefore  $x^x \Big|_0 = \varepsilon^0 = 1.$

The value of a function which takes the form  $0^0$  is usually found, as in the above example, to be unity. This is not, however, *universally* true, as the function

$$\frac{a+x}{x^{\log x}}$$

(one of those earliest adduced for this purpose \*) will show.

This function takes the form  $0^0$ , when  $x = 0$ ; but since its logarithm reduces to  $a + x$ , its value when  $x = 0$  is  $\varepsilon^a$ .

### Examples XIV.

- |                                                        |                        |                                             |
|--------------------------------------------------------|------------------------|---------------------------------------------|
| 1. $(\cos x)^{\cot^2 x},$                              | when $x = 0.$          | $\varepsilon^{-\frac{1}{2}}.$               |
| 2. $\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}},$    | $x = 0.$               | $\frac{1}{\varepsilon}.$                    |
| 3. $(\cos \alpha x)^{\operatorname{cosec}^2 \beta x},$ | $x = 0.$               | $\varepsilon^{-\frac{\alpha^2}{2\beta^2}}.$ |
| 4. $\left(\frac{1}{x}\right)^{\tan x},$                | $x = 0.$               | 1.                                          |
| 5. $(\tan x)^{\tan^2 x},$                              | $x = -\frac{1}{2}\pi.$ | 1.                                          |
| 6. $\left(\frac{1}{x^m}\right)^{x^m} (m > 0),$         | $x = 0.$               | 1.                                          |
| 7. $(1-x)^{\frac{1}{x}},$                              | $x = 0.$               | $\frac{1}{\varepsilon}.$                    |
| 8. $(\sin x)^{\sec^2 x},$                              | $x = \frac{1}{2}\pi.$  | $\varepsilon^{-\frac{1}{2}}.$               |

\* See *Crelle's Journal*, vol. xii, p. 293.

9.  $(\cot x)^{\sin x}, \quad x = 0.$

*Solution:*  $(\cot x)^{\sin x} \Big|_0 = \frac{(\cos x)^{\sin x} \Big|_0}{(\sin x)^{\sin x} \Big|_0} = 1. \quad (\text{See Art. 96.})$

10.  $(\sin x)^{\tan x}, \quad x = 0. \quad 1.$

11.  $(\sin x)^{\tan x}, \quad x = \frac{\pi}{2}. \quad 1.$

12.  $x^{\frac{a}{\log \sin x}}, \quad x = 0. \quad e^a.$

13.  $(\sin x)^{\frac{a^2 - x^2}{\log \tan x}}, \quad x = 0. \quad e^{a^2}.$

14.  $x^{x^a} (a > 0), \quad x = 0. \quad 1.$

15.  $(x^x)^{\frac{(a+x)^2}{\log(x + \log \cos x)}}, \quad x = 0. \quad e^{2a^2},$

16.  $x^{\frac{1}{1-x}}, \quad \text{when } x = 1. \quad \frac{1}{e}.$

17.  $x^{x^2-1}, \quad x = 0. \quad 1.$

18.  $(\cos mx)^{\frac{n}{x^2}}, \quad x = 0. \quad e^{-\frac{1}{2}nm^2}.$

19.  $\left(\frac{\log x}{x}\right)^{\frac{1}{x}}, \quad x = \infty. \quad 1.$

20.  $(1 \pm x)^{\frac{1}{x}}, \quad x = \infty. \quad 1.$

21.  $x^m (\sin x)^{\tan x \left(\frac{\pi - 2x}{2 \sin 2x}\right)^2}, \quad x = \frac{\pi}{2}. \quad \frac{\pi^m}{2^{m+1}}.$

22.  $\frac{(m^2 - 1)(a \sin x - \sin ax)^n}{x^n \sin x (\cos x - \cos ax)^n}, \quad x = 0. \quad \left(\frac{a}{3}\right)^n \log m.$



## CHAPTER VI.

### MAXIMA AND MINIMA OF FUNCTIONS OF A SINGLE VARIABLE.

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#### XV.

#### *Conditions Indicating the Existence of Maxima and Minima.*

**97.** IF, while the independent variable increases continuously, a function dependent on it increases up to a certain value, and then decreases, this value of the function is said to be a *maximum* value. In other words, a function  $f(x)$  has a maximum value corresponding to  $x = a$ , if, when  $x$  increases through the value  $a$ , the function changes from an increasing to a decreasing function.

Since  $f'(x)$  is positive, when  $f(x)$  is an increasing function, and negative when it is a decreasing function; it is obvious that if  $f(a)$  is a maximum value of  $f(x)$ ,  $f'(x)$  must *change sign*, from  $+$  to  $-$ , as  $x$  increases through the value  $a$ .

On the other hand, a function is said to have a *minimum* value for  $x = a$ , if it is a decreasing function before  $x$  reaches this value and an increasing one afterward. In this case,  $f'(x)$  changes sign from  $-$  to  $+$ .

**98.** The derivative  $f'(x)$  can only change sign on passing through zero or infinity. Hence a value of  $x$ , for which  $f(x)$  is a maximum or a minimum, must satisfy one of the two following equations :

$$f'(x) = 0 \quad \text{and} \quad f'(x) = \infty.$$

The required values of  $x$  will therefore be found among the roots of these equations.

The case which usually presents itself, and which will therefore be considered first, is that in which the required value of  $x$  is a root of the equation  $f'(x) = 0$ .

**99.** As an illustration, let it be required *to divide a number into two such parts that the square of one part multiplied by the cube of the other shall give the greatest possible product.*

Denote the given number by  $a$ , and the part to be squared by  $x$ ; then we have

$$f(x) = x^2(a - x)^3.$$

It is evident that a maximum value of this function exists; for when  $x = 0$  its value is zero, and when  $x = a$  its value is again zero, while for intermediate values of  $x$  it is positive; hence the function must change from an increasing to a decreasing function at least once, while  $x$  passes from the value zero to the value  $a$ .

Taking the derivative of this function, the equation

$$f'(x) = 0$$

is in this case  $2x(a - x)^3 - 3x^2(a - x)^2 = 0$ ,

or  $x(a - x)^2(2a - 5x) = 0$ .

0 and  $a$  are roots of this equation; but, as we are in search of a value of the function corresponding to an intermediate value of  $x$ , we put

$$2a - 5x = 0,$$

and obtain

$$x = \frac{2}{5}a.$$

The corresponding value of the function is  $\frac{108}{125}a^5$ , the maximum value sought.

### *Maxima and Minima of Geometrical Magnitudes.*

**100.** When the maximum or minimum value of a geometrical magnitude limited by certain conditions is required, it is necessary to obtain an expression for the magnitude in terms of a single unknown quantity, such that the determination of the value of this quantity will constitute the solution of the problem. For example: *let it be required to determine the cone of greatest convex surface among those which can be inscribed in a sphere whose radius is  $a$ .*

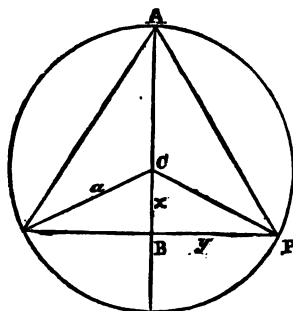


FIG. 10.

Any point  $A$  of the surface of the sphere being taken as the apex of the cone, let the diagram represent a great circle of the sphere passing through the fixed point  $A$ .

If we refer the position of the point  $P$  to rectangular coordinates, and take  $C$  as the origin, the required cone will evidently be determined when  $x$  is determined. We have now to express the convex surface  $S$  in terms of  $x$ .

The expression for the convex surface of a cone gives

$$S = \pi y \sqrt{y^2 + (a + x)^2}, \quad \dots \dots \dots (1)$$

in which the unknown quantities  $x$  and  $y$  are connected by the equation of the circle

$$x^2 + y^2 = a^2. \quad \dots \dots \dots (2)$$

Substituting the value of  $y$ , we have

$$S = \pi \sqrt{a^2 - x^2} \sqrt{2a^2 + 2ax},$$

$$\text{reducing,} \quad S = \pi \sqrt{2a} (a + x) \sqrt{a - x}. \quad \dots \dots \dots (3)$$

Since the factor  $\pi \sqrt{2a}$  is constant, we are evidently required to find the value of  $x$  for which the function

$$f(x) = (a + x) \sqrt{a - x}$$

is a maximum. The equation  $f'(x) = 0$  is, in this case,

$$\sqrt{a - x} - \frac{a + x}{2\sqrt{a - x}} = 0;$$

whence

$$x = \frac{1}{3}a.$$

The altitude of the required cone is therefore  $\frac{2}{3}a$ . Substituting this value of  $x$  in equation (3), we have

$$S = \frac{2}{3} \sqrt{3} \cdot \pi a^2,$$

the maximum value required.

101. As a further illustration, let it be required to determine the greatest cylinder that can be inscribed in a given segment of a paraboloid of revolution.

Let  $a$  denote the altitude, and  $b$  the radius of the base of the segment. The equation of the generating parabola is of the form

$$y^2 = 4cx.$$

Since  $(a, b)$  is a point of the curve, we have the condition

$$b^2 = 4ca;$$

eliminating  $4c$ , the equation of the curve is

$$y^2 = \frac{b^2}{a} x. \quad \dots \dots \dots (I)$$

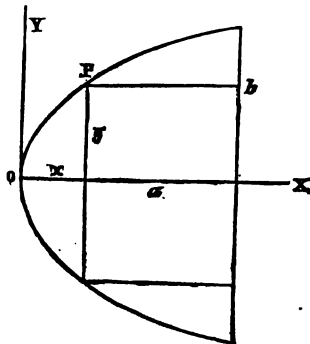


FIG. 11.

The volume  $V$  of the cylinder of which the maximum is required is expressed by

$$V = \pi y^2(a - x),$$

or, by equation (1), 
$$V = \pi \frac{b^2}{a} x(a - x).$$

Hence we put 
$$f(x) = ax - x^2, \quad \therefore f'(x) = a - 2x$$

and the condition  $f'(x) = 0$  gives

$$x = \frac{1}{2}a.$$

Consequently  $a - x$ , the altitude of the cylinder, is one half the altitude of the segment.

### Examples XV.

1. Find the sides of the largest rectangle that can be inscribed in a semicircle of radius  $a$ . The sides are  $a\sqrt{2}$  and  $\frac{1}{2}a\sqrt{2}$ .

2. Determine the maximum right cone inscribed in a given sphere. The altitude is four thirds the radius of the sphere.

3. Determine the maximum rectangle inscribed in a given segment of a parabola.

The altitude of the rectangle is two thirds that of the segment.

4. Find the maximum cone of given slant height  $a$ .

The radius of the base is  $\frac{1}{3}a\sqrt{6}$ .

5. A boatman 3 miles out at sea wishes to reach in the shortest time possible a point on the beach 5 miles from the nearest point of the shore; he can pull at the rate of 4 miles an hour, but can walk at the rate of 5 miles an hour; find the point at which he must land.

*Express the whole time in terms of the distance of the required point from the nearest point of the shore.*

He must land one mile from the point to be reached.

6. If a square piece of sheet-lead whose side is  $a$  have a square cut out at each corner, find the side of the latter square in order that the remainder may form a vessel of maximum capacity

The side of the square is  $\frac{1}{4}a$ .

7. A given weight is to be raised by means of a lever weighing  $n$  pounds per linear inch, which has its fulcrum at one end, and at a fixed distance  $a$  from the point of suspension of the weight  $w$ ; find the length of the lever in order that the power required to raise the weight may be a minimum.

$$\sqrt{\frac{2aw}{n}}$$

8. A rectangular court is to be built so as to contain a given area  $c^2$ , and a wall already constructed is available for one of the sides; find its dimensions so that the least expense may be incurred.

The side parallel to the wall is double each of the others.

9. Determine the maximum cylinder inscribed in a given cone.

The altitude of the cylinder is one third that of the cone.

10. Prove that the rectangle with given perimeter and maximum area is a square, also that the rectangle with given area and minimum perimeter is a square.

11. Find the side of the smallest square that can be inscribed in a square whose side is  $a$ .

*Take as the independent variable the distance between the angles of the two squares.*

$$\frac{1}{2}a\sqrt{2}.$$

12. Inscribe the maximum cone in a given paraboloid, the apex of the cone being at the middle point of the base of the paraboloid.

The altitude of the cone is half that of the paraboloid.

13. Find the maximum cylinder that can be inscribed in a sphere whose radius is  $a$ .

$$\text{The altitude is } \frac{4}{3}a\sqrt{3}.$$

14. Through a point whose rectangular coordinates are  $a$  and  $b$  draw a line such that the triangle formed by this line and the coordinate axes shall be a minimum.

The intercepts on the axes are  $2a$  and  $2b$ .

15. A high vertical wall is to be braced by a beam which must pass over a parallel wall  $a$  feet high and  $b$  feet distant from the other, find the length of the shortest beam that can be used for this purpose.

*Take as the independent variable the inclination of the beam to the horizon*

$$(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}.$$

16. The illumination of a plane surface by a luminous point being directly as the cosine of the angle of incidence of the rays, and inversely as the square of its distance from the point; find the height at which a bracket-burner must be placed, in order that a point on the floor of a room at the horizontal distance  $a$  from the burner may receive the greatest possible amount of illumination.

$$\text{The height is } \frac{a}{\sqrt{2}}.$$

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## XVI.

### *Methods of Discriminating between Maxima and Minima.*

102. When the existence of a maximum or a minimum corresponding to a particular root  $a$  of the equation  $f'(x) = 0$  is not obvious from the nature of the problem, it is necessary to determine whether  $f'(x)$  *changes sign* as  $x$  passes through the value  $a$ .

If a change of sign does take place we have, in accordance with Art. 97, a *maximum* if, when  $x$  passes through the value  $a$ , the change of sign is from  $+$  to  $-$ ; that is, if  $f'(x)$  is a *decreasing* function, and a *minimum* if the change of sign is from  $-$  to  $+$ , in which case  $f'(x)$  is an *increasing* function.

103. In many cases we are able to distinguish maxima from minima by examining the expression for  $f'(x)$ , as in the following examples.

Given 
$$f(x) = \frac{x}{\log x},$$

whence 
$$f'(x) = \frac{\log x - 1}{(\log x)^2};$$

$f'(x) = 0$  gives  $\log x = 1$ , or  $x = e$ .

Since  $\log x$  is an increasing function, it is obvious that, as  $x$  increases through the value  $e$ ,  $f'(x)$  increases; it therefore changes sign from  $-$  to  $+$ , and consequently  $f(e)$  is a minimum value of  $f(x)$ .

104. If  $f'(x)$  does not change sign we have neither a maximum nor a minimum; thus, let

$$f(x) = x - \sin x,$$

whence 
$$f'(x) = 1 - \cos x.$$

In this case  $f'(x)$  becomes zero when  $x = 2n\pi$ ,  $n$  being zero or any integer, but does not change sign, since  $1 - \cos x$  can never be negative; consequently  $f(x)$  has neither maxima nor minima values, but is an *increasing* function for all values of  $x$ .

### *Alternate Maxima and Minima.*

105. Let the curve

$$y = f(x)$$

be constructed, and suppose it to take the form represented in Fig. 12. There is a maximum value of  $f(x)$  at  $B$ , another at  $D$ , and minima values occur at  $A$ , at  $C$ , and at  $E$ .

It is obvious that in a continuous portion of the curve maxima and minima ordinates must occur alternately, and must separate the curve into segments in which the ordinate is alternately an increasing and a decreasing function; hence, if  $f(x)$  has maxi-

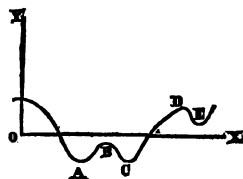


FIG. 12.



ma and minima values, they must occur alternately *unless infinite values of the function intervene*. It is also evident, with the same restriction, that a maximum is greater in value than either of the adjacent minima, but not necessarily greater than *any* other minimum; thus, in Fig. 12, the maximum at *B* is greater than the minima at *A* and *C*, but not greater than that at *E*.

**106.** As an illustration let us take the following function in which it is easy to discriminate between the maxima and minima values.

$$f(x) = x(x+a)^2(x-a)^2.$$

Whence,

$$\begin{aligned} f'(x) &= (x+a)^2(x-a)^2 + 2x(x+a)(x-a)^2 + 3x(x+a)^2(x-a)^2, \\ &= (x+a)(x-a)^2(6x^2 + ax - a^2). \end{aligned}$$

*a* and  $-a$  are evidently roots of  $f'(x) = 0$ ; the roots derived by putting the last factor equal to zero and solving are  $-\frac{1}{2}a$  and  $\frac{1}{3}a$ . Hence  $f'(x)$  can be written in the form

$$f'(x) = 6(x+a)(x+\frac{1}{2}a)(x-\frac{1}{3}a)(x-a)^2,$$

in which the factors are so arranged that the corresponding roots are in order of magnitude.

When  $x < -a$ ,  $f'(x)$  is negative, and, if we regard  $x$  as increasing continuously,  $f'(x)$  changes sign when  $x = -a$ , when  $x = -\frac{1}{2}a$ , and again when  $x = \frac{1}{3}a$ , but *not* when  $x = a$ .

Since  $f'(x)$  is at first negative it changes sign from  $-$  to  $+$  when it first passes through zero, that is when  $x = -a$ ; the corresponding value of  $f(x)$  is therefore a minimum. Accordingly the value of  $f(x)$  corresponding to the next root  $x = -\frac{1}{2}a$  is a maximum, and that corresponding to  $x = \frac{1}{3}a$  is another minimum; but there is neither a maximum nor a minimum corresponding to  $x = a$ .

**107.** When the function is continuous as in the above example, that is, does not become infinite for any finite value of  $x$ , it is always easy to determine by examining the function itself whether the last, or greatest value of  $x$  in question, gives a maximum or a minimum. Thus, in the above example,  $f(x)$  evidently increases without limit as  $x$  increases without limit; therefore, the last value must be a minimum.

*The Employment of a Substituted Function.*

**108.** Since an increasing function of a variable increases and decreases with the variable, such a function will pass from a state of increase to a state of decrease, or the reverse, simultaneously with the variable; that is, it will reach a maximum or a minimum value at the same time with the variable.

This fact often enables us to simplify the determination of maxima and minima by substituting an increasing function of the given function for the given function itself. For example, if we have

$$f(x) = \sqrt{b^2 + ax} + \sqrt{b^2 - ax},$$

we may with advantage employ the square of the given function. The square is

$$2b^2 + 2\sqrt{b^4 - a^2x^2},$$

which is obviously a maximum when  $x = 0$ , and, since the square of a positive quantity is an increasing function, we infer that  $f(x)$  is likewise a maximum for the same value of  $x$ .

**109.** A *decreasing* function of the given function may also be employed; but, in this case, since the substituted function decreases with the increase of the given function and increases with its decrease, a maximum of the substituted function indi-

cates a minimum, and a minimum indicates a maximum of the given function.

Thus, if we have

$$f(x) = \frac{x}{x^3 - 3x + 1},$$

the reciprocal may be employed. The reciprocal of this function is

$$\frac{x^3 - 3x + 1}{x} = x - 3 + \frac{1}{x};$$

whence, taking the derivative, we obtain

$$1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2},$$

which vanishes when  $x = \pm 1$ .

Since  $x^3$  is an increasing function when  $x$  is positive, this derivative is evidently an *increasing* function when  $x = 1$ . The reciprocal is therefore a *minimum* for this value of  $x$ , and consequently  $f(1)$  is a *maximum* value of  $f(x)$ . In a similar manner it may be shown that  $f(-1)$  is a minimum.

### Examples XVI.

Determine the maxima and minima of the following functions :

$$1. f(x) = x^e. \qquad \text{A min. for } x = \frac{1}{e}.$$

$$2. f(x) = \frac{\log x}{x^2}. \qquad \text{A max. for } x = e^{\frac{1}{2}}.$$

$$3. f(x) = \frac{(a-x)^3}{a-2x}. \qquad \text{A min. for } x = \frac{1}{2}a.$$

$$4. f(x) = \frac{1+3x}{\sqrt[4]{1+5x}}. \qquad \text{A min. for } x = -\frac{1}{15}.$$

5.  $f(x) = \sin 2x - x.$

A max. for  $x = n\pi + \frac{1}{2}\pi$ ;  
a min. for  $x = n\pi - \frac{1}{2}\pi.$

6.  $f(x) = 2x^3 + 3x^2 - 36x + 12.$

A max. for  $x = -3$ ;  
a min. for  $x = 2.$

7.  $f(x) = x^3 - 3x^2 - 9x + 5.$

A max. for  $x = -1$ ;  
a min. for  $x = 3.$

8.  $f(x) = 3x^3 - 125x^2 + 2160x.$

A max. for  $x = -4$  and  $x = 3$ ;  
a min. for  $x = -3$  and  $x = 4.$

9.  $f(x) = b + c(x - a)^{\frac{5}{3}}.$

Neither a max. nor a min.

10.  $f(x) = (x - 1)^4 (x + 2)^3.$

A max. for  $x = -\frac{5}{4}$ ;  
a min. for  $x = 1.$

11.  $f(x) = (x - 9)^5 (x - 8)^4.$

A max. for  $x = 8$ ;  
a min. for  $x = 8\frac{1}{2}.$

12.  $f(x) = \frac{1 - x + x^2}{1 + x - x^2}.$

A min. for  $x = \frac{1}{2}.$

13.  $f(x) = \frac{ax}{ax^2 - bx + a}.$  See Art. 109.

Max. for  $x = 1$ ;  
min. for  $x = -1.$

14.  $f(x) = (a^2 - 1)^{n+2}.$

Min. for  $x = -\frac{1}{2}$  ( $a$  being positive).

15.  $f(x) = (1 + x^{\frac{3}{2}})(7 - x)^2.$

Solve by putting  $x = z^2$ . For method of discriminating between maxima and minima, see Art. 107.

Min. for  $x = 0$ , and  $x = 7$ ;  
max. for  $x = 1.$

16.  $f(x) = 5x^5 + 12x^3 - 15x - 40x^2 + 15x^3 + 60x + 27.$

Min. for  $x = -2.$

17.  $f(x) = x^5 - 6x^4 + 4x^3 + 9x^2 - 12x + 3.$

Min. for  $x = -2$ , and  $x = 1$ ;  
max. for  $x = -1.$

18. The top of a pedestal which sustains a statue  $a$  feet in height is  $b$  feet above the level of a man's eyes ; find his horizontal distance from the pedestal when the statue subtends the greatest angle.

When the distance  $= \sqrt{b(a + b)}$ .

19. It is required to construct from two circular iron plates of radius  $a$  a buoy, composed of two equal cones having a common base, which shall have the greatest possible volume.

The radius of the base  $= \frac{1}{3}a\sqrt{6}$ .

20. The lower corner of a leaf of a book is folded over so as just to reach the inner edge of the page ; find when the crease thus formed is a minimum.

*Solution .—*

Let  $y$  denote the length of the crease,  $x$  the distance of the corner from the intersection of the crease with the lower edge, and  $a$  the width of the page.

By means of the relations of similar right triangles, the following expression is deduced :

$$y = \frac{x\sqrt{x}}{\sqrt{x - \frac{1}{2}a}}.$$

Whence we obtain

$$x = \frac{3}{4}a,$$

which gives a minimum value of  $y$ .

21. Find when the area of the part folded over is a minimum.

When  $x = \frac{2}{3}a$ .

## XVII.

### *The Employment of Derivatives Higher than the First.*

110. To ascertain whether  $f'(x)$  is an increasing or a decreasing function, (and thence whether  $f(x)$  is a minimum or a maximum), it is frequently necessary to find the expression for its derivative,  $f''(x)$ . Now, if  $f''(a)$  is found to have a *positive* value, it follows that  $f'(x)$  is an *increasing* function when  $x = a$ ,

and, as was shown in Art. 102, that  $f(a)$  is a minimum. On the other hand, if we find that  $f''(a)$  has a *negative* value, it follows that  $f'(x)$  is a decreasing function, and that  $f(a)$  is a maximum. To illustrate, let

$$f(x) = 3x^4 - 16x^3 - 6x^2 + 12,$$

then 
$$f'(x) = 12x^3 - 48x^2 - 12x.$$

The roots of  $f'(x) = 0$  are  $x = 0$ , and  $x = 2 \pm \sqrt{5}$ .

In this case 
$$f''(x) = 36x^2 - 96x - 12,$$

hence 
$$f''(0) = -12;$$

$f(x)$  is therefore a *maximum* when  $x = 0$ .

It is unnecessary to find the values of  $f''(x)$  for the other roots; for, since the function does not admit of infinite values, the maxima and minima occur alternately. The root  $2 - \sqrt{5}$  being negative and  $2 + \sqrt{5}$  positive, the root zero is intermediate in value, and therefore both the remaining roots give minima.

III. If  $f'(x)$  contains a positive factor which cannot change sign, this factor may be omitted; since we can determine whether  $f'(x)$  increases or decreases through zero by examining the sign of the derivative of the remaining factor. Thus, if

$$f(x) = \frac{x}{1+x^2}, \quad f'(x) = \frac{1-x^2}{(1+x^2)^2}.$$

Since  $\frac{1}{(1+x^2)^2}$  is always positive, we have only to determine whether the factor  $1-x^2$  changes sign. Denoting this factor by  $v$ , and putting  $v = 0$ , we have

$$x = \pm 1.$$

Now 
$$\frac{dv}{dx} = -2x$$

which is negative for  $x = 1$  and positive for  $x = -1$ . These

roots, therefore, give respectively a maximum and a minimum value of  $f(x)$ .

112. There may be roots of the equation  $f'(x) = 0$  which correspond to neither maxima nor minima, since it is a condition essential to the existence of such values that  $f'(x)$  shall change sign. When such cases arise, the form assumed by the curve  $y = f(x)$  in the immediate vicinity of the point at which  $x = a$  will be one of those represented at  $A$  and  $B$  in Fig. 13.

At these points the value of  $\tan \phi$  or  $f'(x)$  is zero, but at  $A$  it is positive on both sides of the point, and  $f(x)$  or  $y$  is an increasing function, while at  $B$   $f'(x)$  is negative on both sides of the point, and  $f(x)$  is a decreasing function.

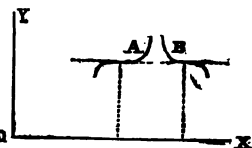


FIG. 13.

113. It is important to notice that at  $A$  the value zero assumed by  $f'(x)$  constitutes a minimum value of this function, thus a root of  $f'(x) = 0$  for which  $f'(x)$  is a *minimum* corresponds to a case in which  $f(x)$  is an *increasing* function. In like manner a root of  $f'(x) = 0$  for which  $f'(x)$  is a *maximum* is a case in which  $f(x)$  is a *decreasing* function.

114. It follows from the preceding article and from Art. 102 that, if  $f'(a) = 0$ , then, of the two functions  $f(x)$  and  $f'(x)$ , one will be a *maximum* and the other a *decreasing* function, or else one will be a *minimum* and the other an *increasing* function. Hence, if we consider the case in which the given function and several of its successive derivatives vanish for the same value of  $x$ , it is evident that when these functions are arranged in order *they will be either alternately maxima and decreasing functions, or alternately minima and increasing functions.*

115. Now suppose that  $f''(x)$  is the first of these successive

derivatives that does not vanish when  $x = a$ , then, writing the series of functions

$$f(x), f'(x), f''(x), \cdot \cdot \cdot \cdot f^{n-1}(x), f^n(x),$$

let us assume first that  $f^n(a)$  is *positive*. Then in the above series of functions  $f^{n-1}(a)$ ,  $f^{n-3}(a)$ , etc., will be increasing functions while  $f^{n-2}(a)$ ,  $f^{n-4}(a)$ , etc., will be minima.

Now whenever  $n$  is *odd*, the original function will belong to the first of these classes and will be an increasing function, while if  $n$  is *even* the original function will belong to the second class and will be a minimum.

On the other hand, if  $f^n(a)$  has a *negative* value, the series of functions will be alternately decreasing functions and maxima; and when  $n$  is *odd*  $f(a)$  will be a decreasing function, but when  $n$  is *even*  $f(a)$  will be a maximum.

Thus we shall have neither maxima nor minima unless the first derivative, which does not vanish when  $x = a$ , is of an *even order*; but when this is the case we shall have a maximum or a minimum according as the value of this derivative is negative or positive.

**116.** The following function presents a case in which the above principle is advantageously employed.

$$f(x) = e^x + e^{-x} + 2 \cos x,$$

$$f'(x) = e^x - e^{-x} - 2 \sin x.$$

Zero is evidently a root of the equation  $f'(x) = 0$ .\* In this case

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\* Zero is the only root of  $f'(x) = 0$  in this example; for

$$f''(x) = \frac{e^{2x} - 2e^x \cos x + 1}{e^x} > \frac{(e^x - 1)^2}{e^x}.$$

$f''(x)$  therefore cannot be negative, hence  $f'(x)$  cannot again assume the value zero.



$$f''(x) = e^x + e^{-x} - 2 \cos x \quad \therefore f''(0) = 0,$$

$$f'''(x) = e^x - e^{-x} + 2 \sin x \quad \therefore f'''(0) = 0,$$

$$f^{(4)}(x) = e^x + e^{-x} + 2 \cos x \quad \therefore f^{(4)}(0) = 4.$$

The fourth derivative being the first that does not vanish, and having a positive value, we conclude that  $x = 0$  gives a minimum value of  $f(x)$ .

### *Infinite Values of the Derivative.*

117. It was shown in Art. 98 that if we have, for  $x = a$ ,

$$f'(x) = \infty,$$

a maximum value will present itself if  $f'(x)$  changes sign from + to -, and a minimum if it changes sign from - to +. It may, however, happen in these cases that the value of  $f(a)$  is also infinite. When  $f(a)$  is finite, the form of the curve

$$y = f(x)$$

in the vicinity of a maximum or a minimum ordinate of this variety is represented at  $A$  and  $B$  in Fig. 14.

As an example, let

$$f(x) = (x^{\frac{1}{2}} - b^{\frac{1}{2}})^{\frac{2}{3}},$$

whence 
$$f'(x) = \frac{2}{3}x^{-\frac{1}{2}}(x^{\frac{1}{2}} - b^{\frac{1}{2}})^{-\frac{1}{3}}.$$

$f'(x)$  is infinite when  $x = 0$  and when  $x = b$ . When  $x = 0$   $f'(x)$  does not change sign, since  $x^{-\frac{1}{2}}$  cannot be negative, but when  $x = b$  it changes sign from - to +; hence  $f(x)$  has a minimum value when  $x = b$ .

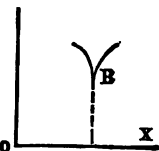


FIG. 14.

## Examples XVII.

1. Show that  $a\epsilon^{bx} + b\epsilon^{-bx}$  has a minimum value equal to  $2\sqrt{ab}$ .

Find the maxima and minima of the following functions :

2.  $f(x) = x \sin x$ .

A maximum for a value of  $x$  in the second quadrant satisfying the equation  $\tan x = -x$ .

3.  $f(x) = \frac{a^2}{x} + \frac{b^2}{a-x}$ .

The roots  $x = \frac{a^2}{a+b}$  and  $x = \frac{a^2}{a-b}$  give a min. and a max. if  $b$  is positive, but a max. and min. if  $b$  is negative.

4.  $f(x) = 2 \cos x + \sin^2 x$ .

*Solution* :—  $f'(x) = 2 \sin x (\cos x - 1)$  ;

rejecting the factor  $2(1 - \cos x)$ , which is always positive, we put

$$v = -\sin x. \quad \text{Hence } \frac{dv}{dx} = -\cos x.$$

A max. for  $x = 2n\pi$  ;

a min. for  $x = (2n+1)\pi$ .

5.  $f(x) = \sin x(1 + \cos x)$ .

A max. for  $x = \frac{1}{3}\pi$  ;

a min. for  $x = -\frac{1}{3}\pi$  ;

neither for  $x = \pi$ ,

6.  $f(x) = \sec x + \log \cos^2 x$ .

*Multiplying the derivative by  $\cos^2 x$ , we obtain*

$$v = \sin x(1 - 2 \cos x).$$

A max. for  $x = 0$ , and  $x = \pi$  ;

a min. for  $x = \pm \frac{1}{3}\pi$ .

7.  $f(x) = \frac{\tan^3 x}{\tan 3x}$ .

A min. for  $x = 0, \frac{2}{3}\pi, \frac{4}{3}\pi$ , and  $\pi$  ;

a max. for  $x = \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi$ , etc.

8.  $f(x) = e^x + e^{-x} - x^2$ .

A min. for  $x = 0$ .

9. Find maxima and minima of the following functions :

$$f(x) = (x^3 - b^3)^{\frac{1}{3}}. \quad \text{A min. for } x = 0.$$

$$10. f(x) = (x^3 - b^3)^{\frac{2}{3}}. \quad \begin{array}{l} \text{A max. for } x = 0; \\ \text{a min. for } x = \pm b. \end{array}$$

$$11. f(x) = (x^3 + 3x + 2)^{\frac{2}{3}} + x^{\frac{2}{3}}.$$

$f'(x) = \infty$  gives min. corresponding to  $x = -2$ ,  $x = -1$  and  $x = 0$ .  
 $f'(x) = 0$  gives two intermediate maxima.

$$12. f(x) = (x^3 + 2x)^{\frac{2}{3}} - (x + 3)^{\frac{2}{3}}. \quad \begin{array}{l} \text{Max. for } x = \frac{1}{4}(-3 \pm \sqrt{17}); \\ \text{min. for } x = 0 \text{ and } x = -2. \end{array}$$

$$13. f(x) = (x - a)^{\frac{2}{3}}(x - b)^{\frac{2}{3}} + c. \quad \begin{array}{l} \text{A max. for } x = \frac{2b + a}{3}; \\ \text{min. for } x = a \text{ and } x = b. \end{array}$$

$$14. f(x) = \frac{(x - a)(x - b)}{x^3}. \quad \text{A min. for } x = \frac{2ab}{a + b}.$$

$$15. f(x) = (x - a)^{\frac{2}{3}}(x - b)^{\frac{2}{3}}.$$

Solutions for  $x = a$  and  $x = \frac{1}{3}(2b + a)$ ; if  $b > a$ , the former gives a max. and the latter a min.

### Miscellaneous Examples.

$$1. f(x) = \frac{x - 1}{x^3 - 3x^2 + 2x + 54}. \quad \text{Use the reciprocal.}$$

Max. for  $x = 4$ .

$$2. f(x) = \frac{x^3 - x + 1}{x^3 + x - 1}. \quad \begin{array}{l} \text{A max. for } x = 0; \\ \text{a min. for } x = 2. \end{array}$$

3.  $f(x) = x^{-a} e^{bx}$ .

A min. for  $x = \frac{a}{b}$ .

4. The equation of the path of a projectile being

$$y = x \tan \alpha - \frac{x^2}{4h \cos^2 \alpha},$$

find the value of  $x$  when  $y$  is a maximum; also the maximum value of  $y$ .  
Max. when  $x = h \sin 2\alpha$ , and  $y = h \sin^2 \alpha$ .

5. In a given sphere inscribe the greatest rectangular parallelopiped.

*Solution :—*

Regarding any one edge as of fixed length, it is easy to show that the other two edges are equal. Hence the three edges are equal.

6. In a given cone inscribe the greatest rectangular parallelopiped.

*Solution :—*

Regarding the parallelopiped as inscribed in a cylinder which is itself inscribed in the cone, the base is evidently a square, and the altitude is that of the maximum cylinder. See Ex. XV, 9.

7. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter, required the height and breadth of the window when the quantity of light admitted is a maximum.

The radius of the semicircle is equal to the height of the rectangle.

8. A tinsmith was ordered to make an open cylindrical vessel of given volume, which should be as light as possible; find the ratio between the height and the radius of the base.

The height equals the radius of the base.

9. What should be the ratio between the diameter of the base and the height of cylindrical fruit-cans in order that the amount of tin used in constructing them may be the least possible?

The height should equal the diameter of the base.

10. Determine the circle having its centre on the circumference of a given circle so that the arc included in the given circle shall be a maximum.

A max. for the value of  $\theta$  which is in the first quadrant.

11. Given the vertical angle of a triangle and its area; find when its base is a minimum. The triangle is isosceles.

12. Prove that, of all circular sectors of the same perimeter, the sector of greatest area is that in which the circular arc is double the radius.

13. Find the minimum isosceles triangle circumscribed about a parabolic segment.

The altitude of the triangle is four-thirds the altitude of the segment.

14. Find the least isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis.

The height is three times the minor semi-axis.

15. Inscribe the greatest parabolic segment in a given isosceles triangle.

The altitude of the segment is three-fourths that of the triangle.

16. A steamer whose speed is 8 knots per hour and course due north sights another steamer directly ahead, whose speed is 10 knots, and whose course is due west. What must be the course of the first steamer to cross the track of the second at the least possible distance from her?

*N.*  $53^{\circ} 8' W.$

17. Determine the angle which a rudder makes with the keel of a ship when its turning effect is the greatest possible.

*Solution :—*

Let  $\phi$  denote the angle between the rudder and the prolongation of the keel of the ship; then if  $b$  is the area of the rudder that of the stream of water intercepted will be  $b \sin \phi$ : the resulting force being decomposed, the component perpendicular to the rudder contains the factor  $\sin^2 \phi$ . Again decomposing this force, and taking the component that is perpendicular to the keel of the ship, which is the only

part of the original force that is effective in turning the ship, the expression to be made a maximum is

$$\sin^2 \phi \cos \phi.$$

Whence we obtain

$$\tan \phi = \sqrt{2}.$$

18. The work of driving a steamer through the water being proportional to the cube of her speed, find her most economical rate per hour against a current running  $a$  knots per hour.

*Solution* :—

Let  $v$  denote the speed of the steamer in knots per hour. The work per hour will then be denoted by  $kv^3$ ,  $k$  being a constant, and the actual distance the steamer advances per hour by  $v - a$ . The work per knot made good is therefore expressed by

$$\frac{kv^3}{v - a}.$$

Whence we obtain the result

$$v = \frac{3}{2}a.$$

## CHAPTER VII.

### THE DEVELOPMENT OF FUNCTIONS IN SERIES.

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#### XVIII.

#### *The Nature of an Infinite Series.*

118. A FUNCTION which can be expressed by means of a limited number of integral terms, involving powers of the independent variable with positive integral exponents only, is called a *rational integral function*.

When  $f(x)$  is *not* a rational integral function, it is usually possible to derive an unlimited series of terms rational and integral with respect to  $x$ , which may be regarded as an algebraic equivalent for the function. The process of deriving this series is called the *development* of the function into an *infinite series*.

When the given function is in the form of a rational fraction, the ordinary process of division (the dividend and divisor being arranged according to ascending powers of  $x$ ) suffices to effect the development. Thus—

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \dots,$$

a series of terms arranged according to ascending powers of  $x$ , each coefficient after the absolute term being 2.

It is to be observed, in the first place, that, owing to the indefinite number of terms in the second member, the equation as written above cannot be verified numerically for an assumed value of  $x$ . In this case, however, the process not

only gives us the series, but the remainder after any number of terms. Thus carrying the quotient to the term containing  $x^n$ , and writing the remainder, we have

$$\frac{1+x}{1-x} = 1 + 2x + 2x^2 + \dots + 2x^n + \frac{2x^{n+1}}{1-x}.$$

This equation may now be verified numerically for any assumed value of  $x$ ; or algebraically by multiplying each member by  $1-x$ , thus obtaining an identity.

The ordinary process of extracting the square root of a polynomial furnishes an example of a series which may be extended so as to include as many terms as we please; but this process gives us no expression for the remainder.

**119.** Assuming that  $f(x)$  admits of development into a series involving ascending powers of  $x$ , and denoting the remainder after  $n+1$  terms by  $R$ , we may write

$$f(x) = A + Bx + Cx^2 + \dots + Nx^n + R, \dots (1)$$

in which  $A, B, C, \dots N$  denote coefficients independent of  $x$ , and as yet unknown; the value of  $R$  is however not independent of  $x$ . If the coefficients  $B, C, \dots N$  admit of finite values, it may be assumed that  $R$  is a function of  $x$  which vanishes when  $x=0$ ; and in accordance with this assumption equation (1) becomes, when  $x=0$ ,

$$f(0) = A, \dots \dots \dots (2)$$

which determines the first term of the series. If in any case the value of  $f(0)$  is found to be infinite, we infer that the proposed development is impossible.

**120.** When the coefficients  $B, C, \dots N$  admit of finite values, and the value of the function to be developed remains



finite,  $R$  will have a finite value. If moreover the value of  $R$  decreases as  $n$  increases, and can be made as small as we please, by sufficiently increasing  $n$ , the series is said to be *convergent*, and may be employed in finding an approximate value of the function  $f(x)$ ; the closeness of the approximation increasing with the number of terms used. A series in which  $R$  does not decrease as  $n$  increases is said to be *divergent*.

When the successive terms of a series decrease it does not necessarily follow that the series is convergent; for the value of the equivalent function, and consequently that of  $R$ , may be infinite. To illustrate, if we put  $x = 1$  in the series

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots,$$

we obtain the numerical series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots;$$

it can be shown that, by taking a sufficient number of terms, the sum of this series may be made to exceed any finite limit, *the value of the equivalent or generating function of the above series being in fact infinite when  $x = 1$ .*\*

121. Since  $R$  vanishes with  $x$ , every series for which finite coefficients can be determined is convergent for certain small values of  $x$ . In some cases there are limiting values of  $x$ , both positive and negative, within which the series is convergent, while for values of  $x$  without these limits the series is divergent. These values of  $x$  are called *the limits of convergence*.

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\* If we consider the first two terms separately, and regard the other terms as arranged in groups of two, four, eight, sixteen, etc., the groups will end with the terms  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ , etc. The sum of the fractions in the first group exceeds  $\frac{1}{2}$  or  $\frac{1}{4}$ , the sum of those in the second exceeds  $\frac{1}{4}$  or  $\frac{1}{8}$ , and so on; hence the sum of  $2N$  such groups exceeds the number  $N$ , and  $N$  may be taken as large as we choose.

The generating function in this case is  $\log \frac{1}{1-x}$ , and unity is the limit of convergence.

We shall now demonstrate a theorem by which a function in the form  $f(x_0 + h)$  may be developed into a series involving powers of  $h$ , and in Section XIX we shall show how this theorem is transformed so as to give the expansion of  $f(x)$  in powers of  $x$ .

### *Taylor's Theorem.*

**122.** A function of  $h$  of the form  $f(x_0 + h)$  in general admits of development in a series involving ascending powers of  $h$ . We therefore assume

$$f(x_0 + h) = A_0 + B_0 h + C_0 h^2 + \dots + N_0 h^n + R_0, \dots (1)$$

in which  $A_0, B_0, C_0, \dots, N_0$  are independent of  $h$ , while  $R_0$  is a function of  $h$  which vanishes when  $h$  is zero. Hence, making  $h = 0$ , we have

$$f(x_0) = A_0.$$

We have now to find the values of  $B_0, C_0, \dots, N_0$ , which are evidently functions of  $x_0$ . For this purpose we put

$$x_1 = x_0 + h, \quad \text{whence} \quad h = x_1 - x_0;$$

substituting, equation (1) takes the form

$$f(x_1) = f(x_0) + B_0(x_1 - x_0) + C_0(x_1 - x_0)^2 + \dots + N_0(x_1 - x_0)^n + R_0,$$

in which we may regard  $x_1$  as constant and  $x_0$  as variable. Replacing the latter by  $x$ , and its functions,  $B_0, C_0, \dots, N_0$ , and  $R_0$ , by  $B, C, \dots, N$ , and  $R$ , we have

$$f(x_1) = f(x) + B(x_1 - x) + C(x_1 - x)^2 + \dots + N(x_1 - x)^n + R. \dots (2)$$

Taking derivatives with respect to  $x$ , we have

$$0 = f'(x) - B + (x_1 - x) \frac{dB}{dx} - 2C(x_1 - x) + (x_1 - x)^2 \frac{dC}{dx} \dots$$

$$- nN(x_1 - x)^{n-1} + (x_1 - x)^n \frac{dN}{dx} + \frac{dR}{dx} \dots \quad (3)$$

To render the development possible,  $B$ ,  $C$ ,  $\dots$   $N$ , and  $R$  must have such values as will make equation (3) *identical*, that is, true for all values of  $x$ .

**123** It is evident that  $B$  may be so taken as to cause the first two terms of equation (3) to vanish, and that, this being done,  $C$  can be so determined as to cause the coefficient of  $(x_1 - x)$  to vanish,  $D$  so as to make the coefficient of  $(x_1 - x)^2$  vanish, and so on. The requisite conditions are

$$f'(x) - B = 0, \quad \frac{dB}{dx} - 2C = 0, \quad \frac{dC}{dx} - 3D = 0, \text{ etc.,}$$

and finally  $(x_1 - x)^n \frac{dN}{dx} + \frac{dR}{dx} = 0.$

From these conditions we derive

$$B = f'(x), \quad C = \frac{1}{2} \frac{dB}{dx} = \frac{1}{2} f''(x),$$

$$D = \frac{1}{3} \frac{dC}{dx} = \frac{1}{1 \cdot 2 \cdot 3} f'''(x), \quad E = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} f^{(4)}(x),$$

and in general  $N = \frac{1}{1 \cdot 2 \dots n} f^{(n)}(x).$

Putting  $x_0$  for  $x$ , and substituting in equation (1) the values of  $A_0$ ,  $B_0$ ,  $C_0$ ,  $\dots$   $N_0$ , we obtain

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0) \frac{h^2}{1 \cdot 2} \dots + f^{(n)}(x_0) \frac{h^n}{1 \cdot 2 \dots n} + R_0. \quad (4)$$

This result is called Taylor's Theorem, from the name of its discoverer, Dr. Brook Taylor, who first published it in 1715.

It is evident from equation (4) that *the proposed expansion is impossible* when the given function or any of its derived functions is infinite for the value  $x_0$ .

### *Lagrange's Expression for the Remainder.*

**124.**  $R$  denotes a function of  $x$  which takes the value  $R_0$  when  $x = x_0$ , and becomes zero when  $x = x_1$ . It has been shown in the preceding article that  $R$  must also satisfy the equation

$$(x_1 - x)^n \frac{dN}{dx} + \frac{dR}{dx} = 0,$$

or, substituting the value of  $N$  determined above,

$$\frac{dR}{dx} = - \frac{(x_1 - x)^n}{1 \cdot 2 \dots n} f^{n+1}(x). \quad . \quad . \quad . \quad . \quad (5)$$

This equation shows that  $\frac{dR}{dx}$  cannot become infinite for any value of  $x$  between  $x_0$  and  $x_1$ , provided  $f^{n+1}(x)$  remains finite and real while  $x$  varies between these limits. Since it follows from the theorem proved in Art. 104 that all preceding derivatives must be likewise finite, the above hypothesis is equivalent to the assumption that  *$f(x)$  and its successive derivatives to the  $(n + 1)$ th inclusive remain finite and real while  $x$  varies from  $x_0$  to  $x_0 + h$ .*

**125.** Let  $P$  denote any assumed function of  $x$  which, like  $R$ , takes the value  $R_0$  when  $x = x_0$  and the value zero when  $x = x_1$ , and whose derivative  $\frac{dP}{dx}$  does not become infinite or imaginary for any value of  $x$  between these limits.

Then,  $R_0$  being assumed to be finite,  $P - R$  denotes a function of  $x$  which vanishes both when  $x = x_0$  and when  $x = x_1$  and whose derivative cannot become infinite for any intermediate value of  $x$ . It follows therefore that the value of this function cannot become infinite for any intermediate value of  $x$ .

Since, as  $x$  varies from  $x_0$  to  $x_1$ ,  $P - R$  starts from the value zero and returns to zero again, without passing through infinity, its numerical value must pass through a maximum; hence its derivative cannot retain the same sign throughout, and as it cannot become infinite it must necessarily become zero for some intermediate value of  $x$ . Since  $x_1 = x_0 + h$  this intermediate value of  $x$  can be expressed by  $x_0 + \theta h$ ,  $\theta$  being a *positive proper fraction*. It is therefore evident that at least one value of  $x$  of the form

$$x = x_0 + \theta h$$

will satisfy the equation

$$\frac{dP}{dx} - \frac{dR}{dx} = 0. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

126. The value of  $P$  will fulfil the required conditions if we assume

$$P = \frac{(x_1 - x)^{n+1}}{h^{n+1}} \cdot R_0,$$

for this function takes the value  $R_0$  when  $x = x_0$  and vanishes when  $x = x_1$ ; moreover its derivative with reference to  $x$ , viz.,

$$\frac{dP}{dx} = - \frac{(n+1)(x_1 - x)^n}{h^{n+1}} \cdot R_0, \quad . \quad . \quad . \quad . \quad (7)$$

does not become infinite for any intermediate value of  $x$ . Substituting in equation (6) the values of the derivatives given in equations (5) and (7), and solving for  $R_0$ , we obtain

$$R_0 = \frac{h^{n+1}}{1 \cdot 2 \cdot \cdot \cdot n \cdot (n+1)} f^{n+1}(x_0 + \theta h). \quad . \quad . \quad . \quad (8)$$

This expression for the remainder was first given by Lagrange.

The series may now be written thus :

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0) \frac{h^2}{1 \cdot 2} \dots$$

$$+ f^n(x_0) \frac{h^n}{1 \cdot 2 \dots n} + f^{n+1}(x_0 + \theta h) \frac{h^{n+1}}{1 \cdot 2 \dots (n+1)} \dots \dots (9)$$

It should be noticed that the above expression for the remainder after  $n + 1$  terms differs from the next, or  $(n + 2)$ th term of the series, simply by the addition of  $\theta h$  to  $x_0$ .

### *The Binomial Theorem.*

127. We shall now apply Taylor's Theorem to the function  $(a + b)^m$  in order to obtain a series involving ascending powers of  $b$ .

In this case  $b$  takes the place of  $h$ , and  $a$  that of  $x_0$ ; hence

$$f(x) = x^m \quad \therefore f(x_0) = x_0^m = a^m$$

$$f'(x) = mx^{m-1} \quad \therefore f'(x_0) = mx_0^{m-1} = ma^{m-1}$$

$$f''(x) = m(m-1)x^{m-2} \quad \therefore f''(x_0) = m(m-1)x_0^{m-2} = m(m-1)a^{m-2}$$

and

$$f^n(x_0) = m(m-1)(m-2) \dots (m-n+1)a^{m-n}.$$

Whence

$$(a + b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{1 \cdot 2} a^{m-2}b^2 \dots$$

$$+ \frac{m(m-1)(m-2) \dots (m-n+1)}{1 \cdot 2 \cdot 3 \dots n} a^{m-n}b^n + \dots$$

This result is called the *Binomial Theorem*.

**Examples XVIII.**

1. To expand  $\log(x_0 + h)$  by Taylor's Theorem.

*Solution* :—

$$f(x_0 + h) = \log(x_0 + h)$$

$$f(x) = \log x \quad \therefore \quad f(x_0) = \log x_0$$

$$f'(x) = \frac{1}{x} \quad \therefore \quad f'(x_0) = \frac{1}{x_0}$$

$$f''(x) = -\frac{1}{x^2} \quad \therefore \quad f''(x_0) = -\frac{1}{x_0^2}$$

$$f'''(x) = \frac{2}{x^3} \quad \therefore \quad f'''(x_0) = \frac{1 \cdot 2}{x_0^3}$$

$$f^{(4)}(x) = -\frac{2 \cdot 3}{x^4} \quad \therefore \quad f^{(4)}(x_0) = -\frac{1 \cdot 2 \cdot 3}{x_0^4}$$

$$\dots \dots \dots$$

$$f^{(n)}(x) = -(-1)^n \frac{1 \cdot 2 \dots (n-1)}{x^n} \quad \therefore \quad f^{(n)}(x_0) = -(-1)^n \frac{1 \cdot 2 \dots (n-1)}{x_0^n}.$$

By substituting in equation (4), Art. 123, we obtain

$$\log(x_0 + h) = \log x_0 + \frac{h}{x_0} - \frac{h^2}{2x_0^2} + \frac{h^3}{3x_0^3} - \frac{h^4}{4x_0^4} \dots - (-1)^n \frac{h^n}{nx_0^n} + R_0.$$

Employing Lagrange's expression for the remainder (Art. 126) we derive

$$R_0 = (-1)^n \frac{h^{n+1}}{(n+1)(x_0 + \theta h)^{n+1}}.$$

2. Expand  $a^{x_0+h}$ .

*Solution* :—

$$f(x_0 + h) = a^{x_0+h}$$

$$\begin{aligned}
 f(x) &= a^x & \therefore f(x_0) &= a^{x_0} \\
 f'(x) &= \log a \cdot a^x & \therefore f'(x_0) &= \log a \cdot a^{x_0} \\
 f''(x) &= (\log a)^2 \cdot a^x & \therefore f''(x_0) &= (\log a)^2 \cdot a^{x_0} \\
 &\dots\dots\dots \\
 f^n(x) &= (\log a)^n \cdot a^x & \therefore f^n(x_0) &= (\log a)^n \cdot a^{x_0}.
 \end{aligned}$$

Substituting in equation (4), Art. 123, we have

$$a^{x_0+h} = a^{x_0} \left[ 1 + \log a \cdot h + (\log a)^2 \frac{h^2}{1 \cdot 2} \dots + \frac{(\log a)^n \cdot h^n}{1 \cdot 2 \dots n} \right] + R_0.$$

3. Find the expansion of  $f(x_0 + h)$ , when  $f(x) = x \log x - x$ , writing the  $(n + 1)^{\text{th}}$  term of the series.

$$\begin{aligned}
 f(x_0 + h) &= x_0 \log x_0 - x_0 + \log x_0 \cdot h + \frac{1}{x_0} \cdot \frac{h^2}{1 \cdot 2} - \frac{1}{x_0^2} \cdot \frac{h^3}{2 \cdot 3} + \dots \\
 &\quad + (-1)^n \frac{1}{x_0^{n-1}} \cdot \frac{h^n}{(n-1)n} \dots
 \end{aligned}$$

4. Expand  $\sin^{-1}(x_0 + h)$  to the fourth term inclusive.

$$\begin{aligned}
 \sin^{-1}(x_0 + h) &= \sin^{-1} x_0 + \frac{h}{(1 - x_0^2)^{\frac{1}{2}}} + \frac{x_0}{(1 - x_0^2)^{\frac{3}{2}}} \cdot \frac{h^2}{1 \cdot 2} \\
 &\quad + \frac{1 + 2x_0^2}{(1 - x_0^2)^{\frac{5}{2}}} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \dots
 \end{aligned}$$

5. Prove that

$$\begin{aligned}
 \sin\left(\frac{1}{2}\pi + h\right) &= \frac{1}{2} \left[ 1 + h\sqrt{3} - \frac{h^2}{1 \cdot 2} - \frac{h^3\sqrt{3}}{1 \cdot 2 \cdot 3} + \frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} \right. \\
 &\quad \left. + \frac{h^5\sqrt{3}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \dots \right]
 \end{aligned}$$

6. Prove that

$$\tan\left(\frac{1}{2}\pi + h\right) = 1 + 2h + 2h^2 + \frac{8}{3}h^3 + \frac{16}{5}h^4 + \dots$$



## XIX.

*Maclaurin's Theorem.*

**128.** We shall now give a particular form of Taylor's Series, which is usually more convenient, when numerical results are to be obtained, than the general form given in the preceding section.

This form of the series is obtained by putting  $x_0 = 0$  and replacing  $h$  by  $x$  in equation (4), Art. 123. Thus,

$$f(x) = f(0) + f'(0)x + f''(0) \frac{x^2}{1 \cdot 2} \cdots + f^{(n)}(0) \frac{x^n}{1 \cdot 2 \cdots n} + R_0 \dots (1)$$

and, the same substitutions being made in equation (8), Art. 126, we obtain

$$R_0 = f^{(n+1)}(\theta x) \frac{x^{n+1}}{1 \cdot 2 \cdots (n+1)}.$$

Equation (1) is called Maclaurin's Theorem: it may be used in developing any function to which Taylor's Theorem is applicable, by giving a different signification to the symbol  $f$ . Thus, if  $\log(1 + h)$  is to be developed by Taylor's Theorem,  $f(x) = \log x$ , the value of  $x_0$  being unity; but, if  $\log(1 + x)$  is to be developed by Maclaurin's Theorem, we must put  $f(x) = \log(1 + x)$ . (Compare Ex. XVIII., 1, with Art. 130.)

*The Exponential Series and the Value of  $e$ .*

**129.** As an example of the application of the above theorem, we shall deduce the development of the function  $e^x$ , which is called the *exponential series*, and shall thence obtain a series for computing the value of  $e$ .

The successive derivatives of  $e^x$  being equal to the original function, the coefficients,  $f(0)$ ,  $f'(0)$ , etc., each reduce to unity;

we therefore derive, by substituting in equation (1) and introducing the value of  $R_0$ ,

$$\epsilon^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} \dots + \frac{x^n}{1 \cdot 2 \dots n} + \epsilon^{\theta x} \cdot \frac{x^{n+1}}{1 \cdot 2 \dots (n+1)}.$$

Putting  $x$  equal to unity, we obtain the following series, which enables us to compute the value of the incommensurable quantity  $\epsilon$  to any required degree of accuracy:

$$\epsilon = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \dots + \frac{1}{1 \cdot 2 \cdot 3 \dots n} + \frac{\epsilon^{\theta}}{1 \cdot 2 \cdot 3 \dots (n+1)}.$$

The computation may be arranged thus, each term being derived from the preceding term by division:

$$\begin{array}{r} 2.5 \\ .1666666667 \\ 4166666667 \\ 833333333 \\ 138888889 \\ 19841270 \\ 2480159 \\ 275573 \\ 27557 \\ 2505 \\ 209 \\ 16 \\ 1 \\ \hline 2.71828182846 \end{array}$$

Since  $\epsilon^{\theta}$  is less than  $\epsilon$ , the remainder ( $n$  being 14) is less than  $\frac{1}{15}$  of the last term employed in the computation, and therefore cannot affect the result. Inasmuch as each term may contain a positive or negative error of one-half a unit in the last decimal

place, we cannot, in general, rely upon the accuracy of the last two places of decimals, in computations involving so large a number of terms. Accordingly, this computation only justifies us in writing

$$e = 2.718281828.$$

### *Logarithmic Series.*

130. The logarithmic series is deduced by applying Mac-laurin's Theorem to the function  $\log(1+x)$ .

In this case

$$f(x) = \log(1+x) \therefore f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \therefore f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \therefore f''(0) = -1$$

$$f'''(x) = \frac{1 \cdot 2}{(1+x)^3} \therefore f'''(0) = 1 \cdot 2$$

$$f^{iv}(x) = -\frac{1 \cdot 2 \cdot 3}{(1+x)^4} \therefore f^{iv}(0) = -1 \cdot 2 \cdot 3,$$

hence  $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (1)$

Since this series is divergent for values of  $x$  greater than unity (see Art. 120), we proceed to deduce a formula for the difference of two logarithms, which may be employed in computing successive logarithms; that is, denoting the numbers corresponding to two logarithms by  $n$  and  $n+h$ , we derive a series for

$$\log(n+h) - \log n = \log \frac{n+h}{n}.$$

A series which could be employed for this purpose might be obtained from (1), by putting  $\frac{n+h}{n}$  in the form  $1 + \frac{h}{n}$ . We obtain, however, a much more rapidly converging series by the process given below.

Substituting  $-x$  for  $x$  in (1), we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \quad (2)$$

Subtracting (2) from (1),

$$\log \frac{1+x}{1-x} = 2 \left[ x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \right], \quad (3)$$

a series involving only the positive terms of series (1).

Putting  $\frac{1+x}{1-x} = \frac{n+h}{n}$ , we derive  $x = \frac{h}{2n+h}$ ; substituting in (3), we have

$$\log \frac{n+h}{n} = 2 \left[ \frac{h}{2n+h} + \frac{1}{3} \frac{h^3}{(2n+h)^3} + \frac{1}{5} \frac{h^5}{(2n+h)^5} + \dots \right]. \quad (4)$$

### *The Computation of Napierian Logarithms.*

131. The series given above enables us to compute Napierian logarithms. We proceed to illustrate by computing  $\log_e 10$ . The approximate numerical value of this logarithm could be obtained by putting  $n = 1$  and  $h = 9$  in (4); but, since the series thus obtained would converge very slowly, it is more convenient first to compute  $\log 2$  by means of the series obtained by putting  $n = 1$  and  $h = 1$  in (4); thus:

$$\log_e 2 = 2 \left[ \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \frac{1}{7} \cdot \frac{1}{3^7} + \dots \right].$$

We then put  $n = 8$  and  $h = 2$  in (4); whence

$$\log_e 10 = 3 \log_e 2 + \frac{2}{3} \left[ \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{3^5} + \frac{1}{7} \cdot \frac{1}{3^7} + \dots \right].$$

In making the computation, it is convenient first to obtain the values of the powers of  $\frac{1}{3}$  which occur in the series for  $\log 2$ , by successive division by 9, and afterwards to derive the values of the required terms of the series by dividing these auxiliary numbers by 1, 3, 5, 7, etc. The same auxiliary numbers are also used in the computation of  $\log_e 10$ . See the arrangement of the numerical work below.

$\frac{1}{3}$	0.3333333333	:	1	0.3333333333
$(\frac{1}{3})^2$	370370370	:	3	123456790
$(\frac{1}{3})^5$	41152263	:	5	8230453
$(\frac{1}{3})^7$	4572474	:	7	653211
$(\frac{1}{3})^9$	508053	:	9	56450
$(\frac{1}{3})^{11}$	56450	:	11	5132
$(\frac{1}{3})^{13}$	6272	:	13	482
$(\frac{1}{3})^{15}$	697	:	15	46
$(\frac{1}{3})^{17}$	77	:	17	5

0.3465735902

2

$$\log_e 2 = 0.6931471804$$

$\frac{1}{3}$	0.3333333333	:	1	0.3333333333
$(\frac{1}{3})^2$	41152263	:	3	13717421
$(\frac{1}{3})^5$	508053	:	5	101611
$(\frac{1}{3})^{13}$	6272	:	7	896
$(\frac{1}{3})^{17}$	77	:	9	9

0.3347153270

0.1115717757

0.2231435513

$$3 \log_e 2 = 2.0794415412$$

$$\log_e 10 = 2.30258509$$

The tabular logarithms of the system of which 10 is the base, are derived from the corresponding Napierian logarithms by means of the relation

$$\log_e x = \log_e 10 \log_{10} x,$$

whence 
$$\log_{10} x = \frac{1}{\log_e 10} \cdot \log_e x = M \cdot \log_e x.$$

The constant  $\frac{1}{\log_e 10}$ , denoted above by  $M$ , is called the *modulus* of common logarithms. Taking the reciprocal of  $\log_e 10$ , computed above, we have

$$M = 0.43429448.$$

### *The Developments of the Sine and the Cosine.*

132. Let 
$$f(x) = \sin x,$$

then

$$f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{(4)}(x) = \sin x;$$

$f^{(4)}$  being identical with  $f$ , it follows that these functions recur in cycles of four; their values when  $x = 0$  are

$$0, 1, 0, -1, \text{ etc.}$$

Hence substituting in equation (1), Art. 128, we have

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots \quad (1)$$

In a similar manner, we obtain

$$\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots \quad (2)$$

**Examples XIX.**

1. Expand
- $(1 + x)^m$
- .

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

It is evident that no coefficient will vanish if  $m$  is negative or fractional. This is the form in which the binomial theorem is employed in computation,  $x$  being less than unity.

2. Find three terms of the expansion of
- $\sin^3 x$
- .

$$\sin^3 x = x^3 - \frac{x^5}{3} + \frac{2x^7}{3^2 \cdot 5} - \dots$$

3. Expand
- $\tan x$
- to the term involving
- $x^5$
- inclusive.

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

4. Expand
- $\sec x$
- to the term involving
- $x^6$
- inclusive.

$$\sec x = 1 + \frac{x^2}{1 \cdot 2} + \frac{5x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{61x^6}{1 \cdot 2 \cdot \dots \cdot 6} + \dots$$

5. Expand
- $\log \sec x$
- to the term involving
- $x^6$
- inclusive.

$$\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots$$

6. Find four terms of the expansion of
- $e^x \sec x$
- .

$$e^x \sec x = 1 + x + x^2 + \frac{2x^3}{3} + \dots$$

7. Derive the expansion of
- $\log(1 - x^2)$
- from the logarithmic series, and verify by adding the expansions of
- $\log(1 + x)$
- and
- $\log(1 - x)$
- .

8. Derive the expansion of
- $(1 + x)e^x$
- from that of
- $e^x$
- .

$$(1 + x)e^x = 1 + 2x + \frac{3 \cdot x^2}{1 \cdot 2} + \dots + \frac{n + 1}{1 \cdot 2 \cdot \dots \cdot n} x^n.$$

9. Find, by means of the exponential series, the expansion of  $x\epsilon^x$ , including the  $n$ th term.

10. Expand  $\frac{\epsilon^x}{1+x}$  by division, making use of the exponential series.

$$\frac{\epsilon^x}{1+x} = 1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{3x^4}{8} - \frac{11x^5}{30} + \dots$$

11. Find the expansion of  $\epsilon^x \log(1+x)$  to the term involving  $x^5$ , by multiplying together a sufficient number of the terms of the series for  $\epsilon^x$  and for  $\log(1+x)$ .

$$\epsilon^x \log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^4}{40} + \dots$$

12. Expand  $\log(1+\epsilon^x)$ .

$$\log(1+\epsilon^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \dots$$

13. Expand  $(1+\epsilon^x)^n$  to the term involving  $x^3$  inclusive.

$$(1+\epsilon^x)^n = 2^n \left\{ 1 + \frac{n}{2} \cdot x + \frac{n(n+1)}{2^2} \cdot \frac{x^2}{1 \cdot 2} + \frac{n(n^2+n+2)}{2^3} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \dots \right\}.$$

14. Find the expansion of  $\sqrt[4]{1 \pm \sin 2x}$ , employing the formula  $\sqrt[4]{1 \pm \sin 2x} = \cos x \pm \sin x$ .

$$\sqrt[4]{1 \pm \sin 2x} = 1 \pm x - \frac{x^2}{1 \cdot 2} \mp \frac{x^3}{1 \cdot 2 \cdot 3} + \dots$$

15. Find the expansion of  $\cos^2 x$  by means of the formula  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ .

$$\cos^2 x = 1 - x^2 + \frac{2^2 x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{2^5 x^6}{1 \cdot 2 \cdot \dots \cdot 6} + \dots$$



16. Find the expansion of  $\cos^3 x$ , by means of the formula  $\cos^3 x = \frac{1}{4}(\cos 3x + 3 \cos x)$ .

$$\cos^3 x = 1 - \frac{3x^2}{1 \cdot 2} + \frac{3^3 + 3}{4} \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} \cdots + (-1)^n \frac{3^{2n} + 3}{4} \cdot \frac{x^{2n}}{1 \cdot 2 \cdots 2n}.$$

17. Compute  $\log_e 3$ , and find  $\log_{10} 3$  by multiplying by the value of  $M$  (Art. 166).

$$\log_e 3 = 1.0986123.$$

$$\log_{10} 3 = 0.4771213.$$

18. Find  $\log_e 269$ .

Put  $n = 270 = 10 \times 3^3$ , and  $h = -1$ .

$$\log_e 269 = 5.5947114.$$

19. Find  $\log_e 7$ , and  $\log_e 13$ .

$$\log_e 7 = 1.9459101.$$

$$\log_e 13 = 2.5649494.$$

## CHAPTER VIII.

### CURVE TRACING.

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#### XX.

#### *Equations in the Form $y = f(x)$ .*

**133.** WHEN a curve given by its equation is to be traced, it is necessary to determine its general form especially at such points as present any peculiarity, and also the nature of those branches of the curve, if there be any, which are unlimited in extent.

The general mode of procedure, when the equation can be put in either of the forms,  $y = f(x)$  or  $x = \phi(y)$ , is indicated in the following examples.

#### *Asymptotes Parallel to the Coordinate Axes.*

**134. Example 1.**  $a^2y - x^2y = a^2$  . . . . . (1)  
Solving for  $y$ , we obtain

$$y = \frac{a^2}{a^2 - x^2} \quad \text{. . . . . (2)}$$

When  $x = 0, y = a$ . Numerically equal positive and negative values of  $x$  give the same values for  $y$ ; the curve is therefore symmetrical with reference to the axis of  $y$ . As  $x$  increases

from zero,  $y$  increases until the denominator,  $a^2 - x^2$ , becomes zero, when  $y$  becomes infinite; this occurs when  $x = \pm a$ .

Draw the straight lines  $x = \pm a$ . These are lines to which the curve approaches indefinitely, for we may assign values to  $x$  as near as we please to  $+a$  or to  $-a$ , thus determining points of the curve as near as we please to the straight lines  $x = a$  and  $x = -a$ . Such lines are called *asymptotes* to the curve.

When  $x$  passes the value  $a$ ,  $y$  becomes negative and decreases numerically, approaching the value zero as  $x$  increases indefinitely. Hence there is a branch of the curve below the axis of  $x$  to which the lines  $x = a$  and  $y = 0$  are asymptotes.

The general form of the curve is indicated in Fig. 15.

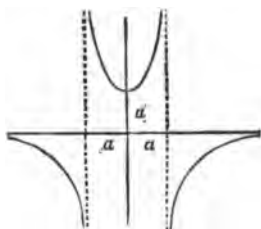


FIG. 15.

The point  $(0, a)$  evidently corresponds to a minimum ordinate.

**135. Example 2.**  $a^2x = y(x - a)^2$ . . . . . (1)  
Solving for  $y$ ,

$$y = \frac{a^2x}{(x - a)^2}. \quad \text{. . . . . (2)}$$

When  $x$  is zero,  $y$  is zero;  $y$  increases as  $x$  increases until  $x = a$ , when  $y$  becomes infinite. Hence

$$x = a$$

is the equation of an asymptote. When  $x$  passes the value  $a$ ,  $y$  does not change sign, but remains positive, and as  $x$  increases  $y$  diminishes, approaching zero as  $x$  increases indefinitely.

Examining now the values of  $y$  which correspond to negative values of  $x$ , we perceive that,  $y$  becoming negative, the branch which passes through the origin continues below the axis of  $x$ , and that  $y$  approaches zero as the negative value of  $x$  increases indefinitely. Hence the general form of the curve is that indicated in Fig. 17.

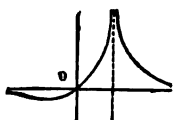


FIG. 17.

**136.** To determine the direction of the curve at any point, we have

$$\tan \phi = \frac{dy}{dx} = -a^2 \frac{a+x}{(x-a)^3} \dots \dots \dots (3)$$

The direction in which the curve passes through the origin is given by the value of  $\tan \phi$  which corresponds to  $x=0$ . From (3), we have

$$\left. \frac{dy}{dx} \right|_0 = 1;$$

the inclination of the curve at the origin is therefore  $45^\circ$ .

### *Minimum Ordinates and Points of Inflexion.*

**137.** To find the minimum ordinate which evidently exists on the left of the axis of  $y$ , we put the expression for  $\frac{dy}{dx}$  equal to zero, and deduce

$$x = -a.$$

The minimum ordinate is therefore found at the point whose abscissa is  $-a$ ; its value, obtained from equation (2), is  $-\frac{1}{4}a$ .

A *point of inflexion* is a point at which  $\frac{d^2y}{dx^2}$  changes sign (see

Art. 74); in other words, it is a point at which  $\tan \phi$  has a maximum or a minimum value. In this case there is evidently a point of inflexion on the left of the minimum ordinate. From equation (3) we derive

$$\frac{d^2y}{dx^2} = 2a^2 \frac{x + 2a}{(x - a)^3},$$

putting this expression equal to zero to determine the abscissa, and deducing the corresponding ordinate from (2), we obtain, for the coordinates of the point of inflexion,

$$x = -2a, \text{ and } y = -\frac{8}{3}a.$$

### *Oblique Asymptotes.*

138. *Example 3.*  $x^3 - 2x^2y - 2x^2 - 8y = 0$ . . . . . (1)

Solving this equation for  $y$ , we have

$$y = \frac{x^3}{2} \cdot \frac{x - 2}{x^2 + 4}. \quad \dots \dots \dots (2)$$

It is obvious from the form of equation (2) that the curve meets the axis of  $x$  at the two points (0, 0) and (2, 0). Since  $y$  is positive only when  $x > 2$ , the curve lies below the axis of  $x$  on the left of the origin, and also between the origin and the point (2, 0), but on the right of this point the curve lies above the axis of  $x$ .

139 Developing the second member of equation (2) into an expression involving a fraction whose numerator is lower in degree than its denominator, we have

$$y = \frac{1}{2}x - 1 + 2 \frac{2 - x}{x^2 + 4}. \quad \dots \dots \dots (3)$$

The fraction in this expression decreases without limit as  $x$  increases indefinitely; hence the ordinate of the curve may, by increasing  $x$ , be made to differ as little as we please from that of the straight line

$$y = \frac{1}{2}x - 1.$$

*This line is, therefore, an asymptote.*

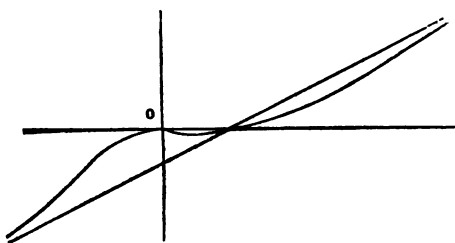


FIG. 18.

The fraction  $\frac{2-x}{x^2+4}$  is positive for all values of  $x$  less than 2, negative for all values of  $x$  greater than 2, and does not become infinite. The curve, therefore, lies above the asymptote on the left of

the point  $(2, 0)$ , and below it on the right of this point, as represented in Fig. 18.

**140.** There is evidently a minimum ordinate between the origin and the point  $(2, 0)$ .

We obtain from equation (2)

$$\frac{dy}{dx} = \frac{x}{2} \cdot \frac{x^2 + 12x - 16}{(x^2 + 4)^2}, \quad \dots \dots (4)$$

and

$$\frac{d^2y}{dx^2} = 4 \cdot \frac{-x^3 + 6x^2 + 12x - 8}{(x^2 + 4)^3}. \quad \dots \dots (5)$$

Putting  $\frac{dy}{dx} = 0$ , we obtain  $x = 0$  and  $x = 1.19$  nearly, the only real roots; the abscissa corresponding to the minimum ordinate is therefore 1.19, the value of the ordinate being about  $-0.11$ . The root zero corresponds to a maximum ordinate at the origin.

Putting  $\frac{d^2y}{dx^2} = 0$ , we obtain the three roots  $x = -2$ , and  $x = 2 (2 \pm \sqrt{3})$ ; the corresponding ordinates are obtained from equation (3). There are, therefore, three points of inflexion, one situated at the point  $(-2, -1)$ , and the others near the points  $(0.54, -0.05)$ , and  $(7.46, 2.55)$ .

The inclination of the curve is determined by means of equation (4) to be  $\tan^{-1}\frac{1}{2}$  at the point  $(2, 0)$ , and  $\tan^{-1}\frac{3}{4}$  at the left-hand point of inflexion.

### *Curvilinear Asymptotes.*

141. *Example 4.*  $x^2 - xy + 1 = 0$ . . . . . (1)  
Solving for  $y$

$$y = \frac{x^2 + 1}{x} = x + \frac{1}{x}. \quad . . . . . (2)$$

In this case, on developing  $y$  in powers of  $x$ , the integral portion of its value is found to contain the second power of  $x$ ; the fraction approaches zero when  $x$  increases indefinitely; hence the ordinate of this curve may be made to differ as little as we please from that of the curve

$$y = x^2. \quad . . . . . (3)$$

The parabola represented by this equation is accordingly said to be a *curvilinear asymptote*. It is indicated by the dotted line in Fig. 19.

142. The sign of the fraction  $\frac{1}{x}$  is always the same as that of  $x$ , and its value is infinite when  $x$  is zero; hence the curve lies below the parabola on the left of the axis of  $y$ , and above it on the right, this axis being an asymptote, as indicated in Fig. 19.

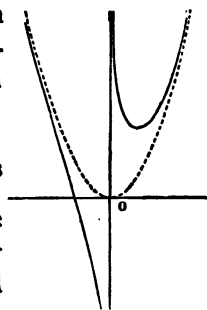


FIG. 19.

Taking derivatives, we obtain

$$\frac{dy}{dx} = 2x - \frac{1}{x^3}, \quad . . . . . (4)$$

and

$$\frac{d^2y}{dx^2} = 2 \left( 1 + \frac{1}{x^3} \right). \quad . . . . . (5)$$

There is a point of inflexion at  $(-1, 0)$ ; the inclination of the curve to the axis of  $x$  at this point is  $\tan^{-1}(-3)$ .

There is a minimum ordinate at the point  $(\frac{1}{2}\sqrt[3]{4}, \frac{3}{2}\sqrt[3]{2})$ .

This cubic curve is an example of the species called by Newton *the trident*. The characteristic property of a trident is the possession of a parabolic asymptote and a rectilinear asymptote parallel to the axis of the parabola.

### Examples XXVI.

1. Trace the curve  $y = x(x^3 - 1)$ .

Since  $y$  is an odd function of  $x$ , the curve is symmetrical with reference to the origin as a centre. Find the point of inflexion, and the minimum ordinate.

2. Trace the curve  $y^3(x - 1) = x^3$ .

The curve has for an asymptote the line  $x = 1$ ; there is a minimum ordinate at  $(2, 2)$ , and a point of inflexion at  $(4, \frac{4}{3}\sqrt[3]{3})$ .

3. Trace the curve  $y^3 = x^3(x - a)$ , determining its direction at the points at which it meets the axis of  $x$ .

The asymptote is found by the method of development, thus

$$y = x \left( 1 - \frac{a}{x} \right)^{\frac{1}{3}} = x \left( 1 - \frac{a}{3x} - \frac{a^2}{9x^2} - \text{etc.} \right);$$

the equation of the asymptote is therefore

$$y = x - \frac{1}{3}a,$$



4. Trace the curve  $x^3 + xy + 2x - y = 0$ .

5. Trace the curve  $y^3 = x^3 + x^2$ , and find its direction at the origin.

The curve has a maximum ordinate at  $(-\frac{2}{3}, \pm \frac{2}{3}\sqrt{3})$ . The value of  $y^3$  may be taken as the function whose maximum is required. (See Art. 108.)

6. Trace the curve  $y = x^3 - xy$ . Find the point of inflexion, the minimum ordinate, and the asymptotes.

The curve has a rectilinear asymptote  $x = -1$ , and a curvilinear asymptote  $y = x^3 - x + 1$ . This curve is a trident. (See Art. 142.)

7. Trace the curve  $y^3 = x^3 - x^4$ .

Both branches of the curve are tangent to the axis of  $x$  at the origin.

8. Trace the curve  $y^3 - 4y = x^3 + x^2$ .

Solving for  $y$ , we obtain

$$y = 2 \pm \sqrt[3]{(x^3 + x^2 + 4)} = 2 \pm \sqrt[3]{(x+2)(x^2 - x + 2)}.$$

The factor  $x^2 - x + 2$  being always positive, the curve is real on the right of the line  $x = -2$ .

Find the points at which the curve cuts the axis, and show that the upper branch has a maximum ordinate for  $x = -\frac{2}{3}$  and a minimum ordinate for  $x = 0$ .

9. Trace the curve  $(x - 2a)xy = a(x - a)(x - 3a)$ .

10. Trace the curve  $(x - 2a)xy^3 = a^3(x - a)(x - 3a)$ .

11. Trace the curve  $y^3 = x^4(1 - x^2)^3$ : find all the points at which the tangent is parallel to the axis of  $x$ .

12. Trace the curve  $6x(1 - x)y = 1 + 3x$ .

This curve has a point of inflexion, determined by a cubic having only one real root, which is between  $-1$  and  $-2$ . Find the three asymptotes, and the maximum and the minimum ordinate.

13. Trace the curve  $x^2y^2 = (x+2)^2(1+x^2)$ .

Solving the equation for  $y$ , we have

$$y = \pm \frac{x+2}{x} \sqrt{(1+x^2)} = \pm (2+x) \left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}}.$$

The asymptotes are  $y = x + 2$ ,  $y = -x - 2$ , and  $x = 0$ . The curve has a minimum ordinate corresponding to  $x = \sqrt[3]{2}$ ; the inclination at the point at which it cuts the axis of  $x$  is  $\tan^{-1}(\pm \frac{1}{2}\sqrt{5})$ . There is a point of inflexion corresponding to the abscissa  $x = -6.1$  nearly.

## XXI.

### *Curves Given by Polar Equations.*

143. The following examples will illustrate some of the methods employed when the curve is given by means of its polar equation.

*Example 5.*  $r = a \cos \theta \cos 2\theta$ . . . . . (I)

When  $\theta = 0$ ,  $r = a$ , the generating point  $P$  therefore starts from  $A$  on the initial line. As  $\theta$  increases,  $r$  decreases and becomes zero when  $\theta = 45^\circ$ ,  $P$  describing the half-loop in the first quadrant, and arriving at the pole in a direction having an inclination of  $45^\circ$  to the initial line. When  $\theta$  passes  $45^\circ$ ,  $r$  becomes negative, and returns to zero again when  $\theta = 90^\circ$ ,  $P$  describing the loop in the third quadrant. As  $\theta$  passes  $90^\circ$ ,  $r$  again becomes positive, but returns to zero when  $\theta = 135^\circ$ ,  $P$  describing the loop in the second quadrant. As  $\theta$  varies from  $135^\circ$  to  $180^\circ$ ,  $r$  again becomes negative,  $P$  describing the half-loop in the fourth quadrant, and returning to  $A$ .

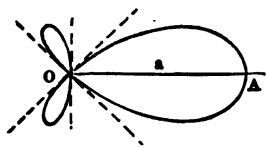


FIG. 19.

In this example if we suppose  $\theta$  to vary from  $180^\circ$  to  $360^\circ$ ,  $P$  will again describe the same curve, and, since  $\theta$  enters the equation of the curve, by means of trigonometrical functions only, it is unnecessary to consider values of  $\theta$  greater than  $360^\circ$ .

144. Putting equation (1) in the form

$$r = a(2 \cos^3 \theta - \cos \theta),$$

we derive

$$\frac{dr}{d\theta} = a(-6 \cos^2 \theta \sin \theta + \sin \theta).$$

To determine the maxima values of  $r$ , we place this derivative equal to zero, thus obtaining the roots

$$\sin \theta = 0 \quad \text{and} \quad \cos \theta = \pm \frac{1}{2} \sqrt{6};$$

the former gives the point  $A$  on the initial line, and the latter gives the values of  $\theta$  which determine the position of the maxima in the small loops. The corresponding values of  $r$  are  $\mp \frac{a}{9} \sqrt{6}$ .

To determine the position of the maximum ordinate, we have from (1)

$$y = r \sin \theta = \frac{1}{2} a \sin 4\theta.$$

The maxima values occur when  $\sin 4\theta = 1$ , and the minima when  $\sin 4\theta = -1$ ; that is, we have maxima when  $\theta = \frac{1}{4}\pi$  and when  $\theta = \frac{5}{4}\pi$ , and minima when  $\theta = \frac{3}{4}\pi$  and  $\frac{7}{4}\pi$ .

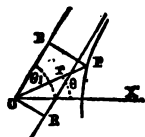
145. In the preceding example the substitution of  $\theta + \pi$  for  $\theta$  changes the sign but not the numerical value of  $r$ . When this is the case,  $\theta$  and  $\theta + \pi$  evidently give the same point of the curve, and the complete curve is described while  $\theta$  varies from 0 to  $\pi$ . If however this substitution changes neither the numerical value nor the sign of  $r$ ,  $\theta$  and  $\theta + \pi$  will give points symmetrically situated with reference to the pole; that is, the curve will be symmetrical in opposite quadrants.

Again if the substitution of  $-\theta$  for  $\theta$  does not change the value of  $r$ ,  $\theta$  and  $-\theta$  give points symmetrically situated with reference to the initial line, hence in this case the curve is symmetrical to this line; but, if the substitution of  $-\theta$  for  $\theta$  changes the sign of  $r$  without changing its numerical value, the curve is symmetrical with reference to a perpendicular to the initial line.

### *The Determination of Asymptotes by Means of Polar Equations.*

146. When  $r$  becomes infinite for a particular value of  $\theta$  the curve has an infinite branch, and, if there be a corresponding asymptote, it may be determined by means of the expression derived below.

Let  $\theta_1$  denote a value of  $\theta$  for which  $r$  is infinite, and let  $OB$  be drawn through the pole, making this angle with the initial line; then, from the triangle  $OBP$ , Fig. 20, we have



$$PB = r \sin (\theta_1 - \theta).$$

FIG. 20. Now, if the curve has an asymptote parallel to  $OB$ , it is plain that as  $\theta$  approaches  $\theta_1$  the limiting value of  $PB$  will be equal to  $OR$ , the perpendicular from the pole upon the asymptote. Hence, if the curve has an asymptote in the direction  $\theta_1$ , the expression

$$OR = [r \sin (\theta_1 - \theta)]_{\theta_1},$$

which takes the form  $\infty \cdot 0$ , will have a finite value, and this value will determine the distance of the asymptote from the pole. Fig. 20 shows that when the above expression is positive  $OR$  is to be laid off in the direction  $\theta_1 - 90^\circ$ .

If upon evaluation the expression for  $OR$  is found to be in-

finite we infer that the infinite branch of the curve is parabolic.

147. *Example 6.*  $r = \frac{a\theta^2}{\theta^2 - 1} \dots \dots \dots (1)$

Since  $r$  becomes infinite when  $\theta = 1$ , we proceed to apply the method established in the preceding article for determining the existence of an asymptote. In this case we have

$$[r \sin(\theta_1 - \theta)]_{\theta_1} = \left[ \frac{a\theta^2}{\theta^2 - 1} \cdot \frac{\sin(1 - \theta)}{\theta - 1} \right]_1 = -\frac{1}{2}a.$$

The angle  $\theta = 1$  corresponds to  $57^\circ 18'$ , nearly, and since the expression for the perpendicular on the asymptote is negative its direction is  $\theta_1 + 90^\circ = 147^\circ 18'$ ; consequently, the asymptote is laid off as in Fig. 21.

Numerically equal positive and negative values of  $\theta$  give the same values for  $r$ ; hence the curve is symmetrical with reference to the initial line.

While  $\theta$  varies from 0 to 1,  $r$  is negative and varies from 0 to  $\infty$ , giving the infinite branch in the third quadrant.

As  $\theta$  passes the value unity, and increases indefinitely,  $r$  becomes positive and decreases, approaching indefinitely to the limiting value  $a$ , which we obtain from (1) by making  $\theta$  infinite. Hence the curve describes an infinite number of whorls approaching indefinitely to the circle  $r = a$ , which is therefore called an *asymptotic circle*.

The points of inflexion in this curve are determined in Art. 175.

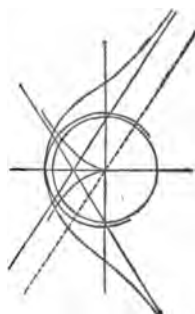


FIG. 21.

**Examples XXI.**

1. Trace the curve  $r = a \cos^{\frac{1}{3}} \theta$ .

Show that, to describe the curve,  $\theta$  must vary from 0 to  $3\pi$ ; also that the curve is symmetrical to the initial line. Find the values of  $\theta$  which correspond to the maxima and minima ordinates and abscissas, the initial line being taken as the axis of  $x$ .

2. Trace the curve  $r = a(2 \sin \theta - 3 \sin^3 \theta)$ .

Show that the entire curve is described while  $\theta$  varies from 0 to  $\pi$ , and that the curve is symmetrical with reference to a perpendicular to the initial line.

3. Trace the curve  $r = 2 + \sin 3\theta$ .

A maximum value of  $r$  (equal to 3) occurs at  $\theta = 30^\circ$ ; a minimum (equal to 1) at  $\theta = 90^\circ$ . The curve is symmetrical with reference to lines inclined at the angles  $30^\circ$ ,  $90^\circ$ , and  $150^\circ$  to the initial line.

4. Trace the curve  $r = 1 + \sin 5\theta$ .

The curve consists of five equal loops.

5. Trace the curve  $r^3 = a^3 \sin 3\theta$ .

The curve consists of three equal loops.

6. Trace the curve  $r \cos \theta = a \cos 2\theta$ .

The curve has an asymptote perpendicular to the initial line at the distance  $a$  on the left of the pole.

7. Trace the curve  $r = 2 + \sin \frac{2}{3}\theta$ .

A maximum value of  $r$  occurs at  $\theta = 60^\circ$ , and a minimum at  $\theta = 180^\circ$ . The curve has three double points, one being on the initial line.

8. Trace the curve  $r \cos 2\theta = a$ .

The curve is symmetrical with reference to the initial line and with reference to a perpendicular to the initial line. There are four asymptotes.

9. Trace the curve  $r \sin 4\theta = a \sin 3\theta$ .

The curve is symmetrical to the initial line, and has three asymptotes; the minimum value of  $r$  is  $\frac{3}{4}a$ .

10. Trace the curve  $r^2 = a^2 \cos 2\theta$ .

The curve is symmetrical with respect to the pole since  $r = \pm a \sqrt{\cos 2\theta}$ :  $r$  is imaginary for values of  $\theta$  between  $\frac{1}{2}\pi$  and  $\frac{3}{2}\pi$ .

11. Trace the curve  $r^{\frac{2}{3}} = a^{\frac{2}{3}} \cos \frac{2}{3}\theta$ .

The curve consists of three equal loops,  $r$  being real for all values of  $\theta$ .

12. Trace the curve  $r^3 \cos \theta = a^3 \sin 3\theta$ .

The curve consists of two loops and an infinite branch which has an asymptote perpendicular to the initial line and passing through the pole.

13. Trace the curve  $r = a \frac{2\theta}{2\theta - 1}$ .

Find the rectilinear and the circular asymptote, and also the point of inflexion.

## XXII.

### *The Parabola of the $n$ th Degree.*

148. The term *parabola* is frequently applied to any curve in which one of the coordinates is proportional to the  $n$ th power of the other,  $n$  being greater than unity. The parabola proper is thus distinguished as the parabola of the second degree.

The general equation of the parabola of the  $n$ th degree is usually written in the homogeneous form, ( $a$  being positive)

$$a^{n-1}y = x^n.$$

The curve passes through the origin and through the point  $(a, a)$ , for all values of  $n$ . Since  $n > 1$ , the curve is tangent to the axis of  $x$  at the origin.

149. The following three diagrams represent forms which the curve takes for different values of  $n$ . When  $n$  denotes a fraction, it is supposed to be reduced to its lowest terms.

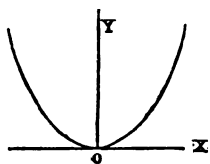


FIG. 22.

Fig. 22 represents the general shape of the curve when  $n$  is an even integer, or a fraction having an even numerator and an odd denominator.

Fig. 23 represents the form of the curve when  $n$  is an odd integer or a fraction with an odd numerator and an odd denominator,

the origin being a point of inflexion.

Fig. 24 represents the form of the curve when  $n$  is a fraction having an odd numerator and an even denominator. In this case  $y$  is regarded as a two-valued function, and is imaginary when  $x$  is negative.

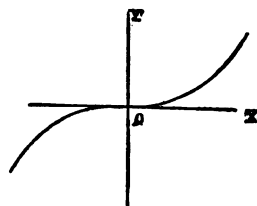


FIG. 23.

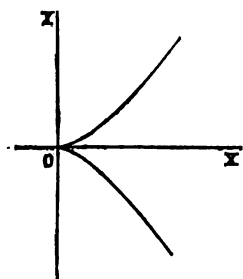


FIG. 24.

Fig. 22 is constructed for the parabola in which  $n = 4$ .

Fig. 23 is the *cubical parabola* in which  $n = 3$ .

Fig. 24 is the *semi-cubical parabola* in which  $n = \frac{3}{2}$ ; the equation being

$$a^{\frac{1}{2}}y = \pm x^{\frac{3}{2}},$$

or

$$ay^2 = x^3.$$

The curves corresponding to the general equation

$$y = A + Bx + Cx^2 + Dx^3 + \dots Lx^n$$

are sometimes called *parabolic curves* of the  $n$ th degree.



*The Cissoid of Diocles.*

**150.** Let  $A$  be a point on the circumference of a circle, and  $BC$  a tangent at the opposite extremity of the diameter  $AB$ ; let  $AC$  be any straight line through  $A$ , and take  $CP = AD$ ; then the locus of  $P$  is the *cissoid*.

To find the polar equation,  $AB$  being the initial line, let  $DB$  be drawn, and denote the radius of the circle by  $a$ ; then  $AC = 2a \sec \theta$ ; and since  $ADB$  is a right angle,  $AD = 2a \cos \theta$ . The polar equation of the locus of  $P$ ,  $A$  being the pole, is, therefore,

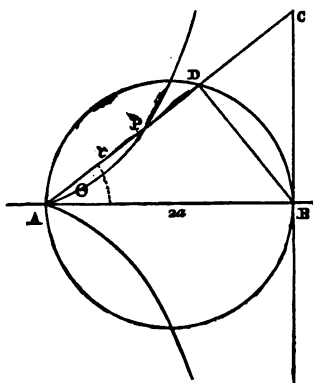


FIG. 25.

$$r = 2a (\sec \theta - \cos \theta) = 2a \frac{1 - \cos^2 \theta}{\cos \theta},$$

or 
$$r = 2a \frac{\sin^2 \theta}{\cos \theta}. \quad \dots \dots \dots (1)$$

**151.** To obtain the rectangular equation, we employ the equations of transformation

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r}, \quad r^2 = x^2 + y^2;$$

whence, eliminating  $\theta$  we obtain

$$r = 2a \frac{y^2}{rx},$$

and thence the rectangular equation of the curve

$$x(x^2 + y^2) = 2ay^2, \quad \dots \dots \dots (2)$$

or 
$$y^2 = \frac{x^3}{2a - x}. \quad \dots \dots \dots (3)$$

### *The Cardioid.*

152. The curve defined by the polar equation

$$r = 2a(1 - \cos \theta) \quad \dots (1)$$

is called the *cardioid*. In Fig. 26, *A* denotes the pole.

The polar equation can also be written in the form

$$r = 4a \sin^2 \frac{1}{2} \theta \quad \dots (2)$$

Transforming equation (1) to rectangular coordinates, we have for the rectangular equation of the cardioid,

$$(x^2 + y^2)^2 + 4ax(x^2 + y^2) - 4a^2y^2 = 0 \quad \dots (3)$$

A point at which two branches of a curve have a common tangent is called a *cusp*. This curve has a cusp at the origin.

### *The Lemniscata of Bernoulli.*

153. The curve defined by the polar equation

$$r^2 = a^2 \cos 2\theta \quad \dots (1)$$

is called the *lemniscata*. In Fig. 27 *O* denotes the pole: *a* is the semi-axis of the curve.

From (1), we have

$$r^2 = a^2 (\cos^2 \theta - \sin^2 \theta),$$

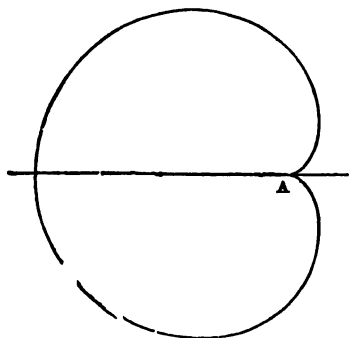


FIG. 26.

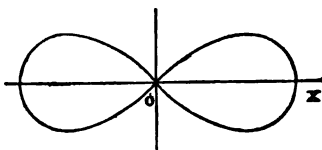


FIG. 27.

or 
$$r^2 = a^2 \frac{x^2 - y^2}{r^2};$$

whence we have

$$(x^2 + y^2)^2 + a^2(y^2 - x^2) = 0, \quad \dots \dots (2)$$

the rectangular equation of the lemniscata, referred to its centre and axis of symmetry.

If we turn the initial line back through  $45^\circ$ , (1) becomes

$$r^2 = a^2 \sin 2\theta, \quad \dots \dots (3)$$

and the corresponding rectangular equation is

$$(x^2 + y^2)^2 = 2a^2 xy. \quad \dots \dots (4)$$

When the equation has this form, the coordinate axes are the tangents at the origin.

### *The Logarithmic or Equiangular Spiral.*

154. This spiral is defined by the polar equation

$$r = ae^{n\theta}, \quad \dots \dots (1)$$

or  $\log r = \log a + n\theta,$

the logarithm of the radius vector being a linear function of the vectorial angle.

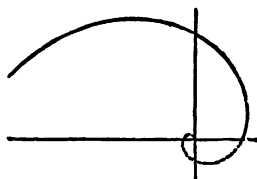


FIG. 28.

It is proved in Art. 168 that this curve cuts its radius vector at a *constant angle* whose cotangent is  $n$ ; hence it is sometimes called the *equiangular spiral*.

### *The Loxodromic Curve.*

155. The track of a ship whose course is uniform is a curve that cuts the meridians of the sphere at a constant angle, and is called a *loxodromic curve*.

If we project this curve stereographically upon the plane of the equator the meridians will project into straight lines, and, since in this projection angles are unchanged in magnitude, the projection of the curve will make a constant angle with the projections of the meridians and will therefore be an equiangular spiral.

Let  $\theta$  denote the longitude of the generating point measured from the point at which the curve cuts the equator, and  $C$  the course; that is, the constant acute angle at which the curve cuts the meridians, the generating point being supposed to approach the pole as  $\theta$  increases. Taking as the pole the projection of the pole of the sphere, the polar equation of the projected curve will be of the form

$$r = ae^{n\theta}, \quad \dots \dots \dots (1)$$

in which  $a$  is the radius of the sphere, since  $\theta = 0$  gives  $r = a$ ; we also have

$$n = -\cot C, \quad \dots \dots \dots (2)$$

since the angle whose cotangent is  $n$  is the supplement of  $C$  (see the preceding article).

Denoting by  $\phi$  the co-latitude of the projected point we have, by the mode of projection,

$$\frac{r}{a} = \tan \frac{1}{2}\phi; \quad \dots \dots \dots (3)$$

and, denoting the corresponding latitude by  $L$ ,

$$\frac{1}{2}\phi = \frac{1}{2}\pi - \frac{1}{2}L.$$

Equation (1) is therefore equivalent to

$$\tan \left( \frac{1}{2}\pi - \frac{1}{2}L \right) = e^{-\theta \cot C};$$

whence, solving for  $\theta$ , we have

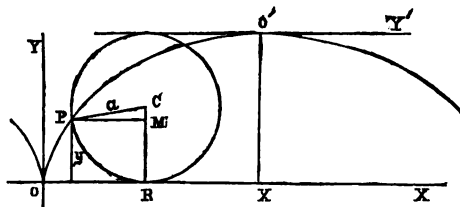
$$\theta = -\tan C \log_e \tan (\tfrac{1}{4}\pi - \tfrac{1}{2}I) = \tan C \log_e \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}I),$$

or, employing common logarithms and expressing  $\theta$  in degrees,

$$\theta^{\circ} = 131.9284 \tan C \log_{10} \tan (45^{\circ} + \frac{1}{2}I) . . . (4)$$

### *The Cycloid.*

**156.** The path described by a point in the circumference of a circle which rolls upon a straight line is called a *cycloid*. The curve consists of an unlimited number of branches cor-



**FIG. 29.**

responding to successive revolutions of the generating circle ; a single branch is, however, usually termed a cycloid.

Let  $O$ , the point where the curve meets the straight line, be taken as the origin, let  $P$  be the generating point of the curve, and denote the angle  $PCR$  by  $\psi$ . Since  $PR$  is equal to the line  $OR$  over which it has rolled,

$$OR = PR = a\psi,$$

and, from Fig. 29, we readily derive

$$\left. \begin{aligned} x &= a(\psi - \sin \psi) \\ y &= a(1 - \cos \psi) \end{aligned} \right\} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (I)$$

**157.** These two equations express the values of  $x$  and  $y$  in terms of the auxiliary variable  $\psi$ , and constitute the equations of the cycloid. If desirable,  $\psi$  is easily eliminated from equations (1) and an equation between  $x$  and  $y$  obtained. Thus, from the second equation, we have

$$\cos \psi = \frac{a - y}{a}, \text{ whence } \sin \psi = \frac{\sqrt{(2ay - y^2)}}{a};$$

and hence from the first of equations (1)

$$x = a \cos^{-1} \frac{a - y}{a} - \sqrt{(2ay - y^2)}, \quad . \quad . \quad . \quad (2)$$

or 
$$x = a \operatorname{vers}^{-1} \frac{y}{a} - \sqrt{(2ay - y^2)}.$$

Equations (1) will in general be found more convenient than equation (2). Thus we easily derive from (1)

$$\frac{dy}{dx} = \frac{\sin \psi \, d\psi}{(1 - \cos \psi) \, d\psi} = \frac{\sin \psi}{1 - \cos \psi};$$

whence

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\cos \psi - 1}{(1 - \cos \psi)^2} \frac{d\psi}{dx} = - \frac{1}{a(1 - \cos \psi)^2}.$$

**158.** The cycloid is frequently referred to the middle point  $O'$  or vertex of the curve as an origin, the directions of the axes being turned through  $90^\circ$ .

Denoting the coordinates referred to the axes  $O'X'$  and  $O'Y'$ , in Fig. 29, by  $x'$  and  $y'$ , we have

$$\begin{aligned} y' &= x - a\pi = a(\psi - \pi - \sin \psi), \\ x' &= 2a - y = a(1 + \cos \psi), \end{aligned}$$

or, denoting  $\psi - \pi$  by  $\psi'$ ,

$$\left. \begin{aligned} y' &= a(\psi' + \sin \psi') \\ x' &= a(1 - \cos \psi') \end{aligned} \right\} . \quad . \quad . \quad . \quad (3)$$

In these equations  $\psi' = 0$  gives the coordinates of the vertex and  $\psi' = \pm \pi$  gives those of the cusps.

*The Epicycloid.*

159. When a circle, tangent to a fixed circle externally, rolls upon it, the path described by a point in the circumference of the rolling circle is called an *epicycloid*.

Taking the origin at the centre of the fixed circle, and the axis of  $x$  passing through  $A$ , (one of the positions of  $P$  when in contact with the fixed circle,)  $a, b, \psi$ , and  $\chi$ , being defined by the diagram, we have, evidently,

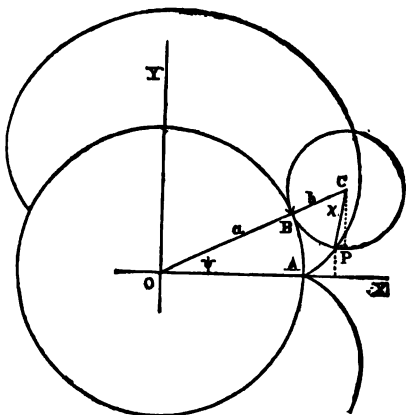


FIG. 30.

$$a\psi = b\chi \therefore \chi = \frac{a}{b}\psi.$$

The inclination of  $CP$  to the axis of  $x$  is equal to  $\psi + \chi$ , or to  $\frac{a+b}{b}\psi$ ; the coordinates of  $P$  are found by subtracting the projections of  $CP$  on the axes from the corresponding projections of  $OC$ ; hence

$$\left. \begin{aligned} x &= (a+b) \cos \psi - b \cos \frac{a+b}{b} \psi \\ y &= (a+b) \sin \psi - b \sin \frac{a+b}{b} \psi \end{aligned} \right\} \quad \dots \quad (I)$$

These are the equations of an epicycloid referred to an axis passing through one of the cusps,

Were the generating point taken at the opposite extremity of a diameter passing through  $P$  in the figure, the projection of  $CP$  would be *added* to that of  $OC$ ; the axis of  $x$  would in this case pass through one of the *vertices* of the curve, and the second terms in the above values of  $x$  and  $y$  would have the *positive sign*.

### The Hypocycloid.

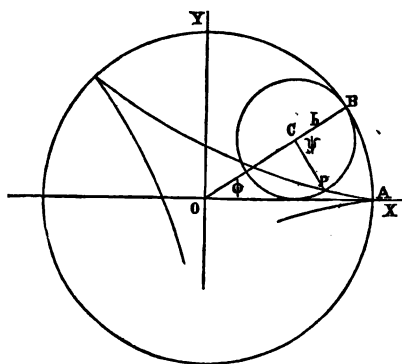


FIG. 31.

160. When the rolling circle has internal contact with the fixed circle, the curve generated by a point on the circumference is called the *hypocycloid*, whether the radius of the rolling circle be greater or less than that of the fixed circle.

Adopting the notation used in deducing the equation of the epicycloid we have (see Fig. 31),

$$OC = a - b, \quad \text{and} \quad \chi = \frac{a}{b}\psi.$$

The inclination of  $CP$  to the negative direction of the axis of  $x$  is

$$\chi - \psi = \frac{a - b}{b}\psi;$$

hence the equations of the *hypocycloid* are

$$\left. \begin{aligned} x &= (a - b) \cos \psi + b \cos \frac{a - b}{b} \psi \\ y &= (a - b) \sin \psi - b \sin \frac{a - b}{b} \psi \end{aligned} \right\} \dots \dots (1)$$



*The Four-Cusped Hypocycloid.*

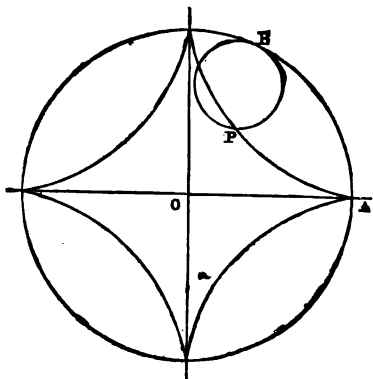


FIG. 32.

161. In the case of the hypocycloid when  $b = \frac{1}{4}a$ , the circumference of the rolling circle is one-fourth the circumference of the fixed circle, and the curve will have a cusp at each of the four points where the coordinate axes cut the fixed circle, as represented in Fig. 32.

On substituting  $\frac{1}{4}a$  for  $b$  equations (2) Art. 160 become

$$\left. \begin{aligned} x &= \frac{3}{4}a \cos \psi + \frac{1}{4}a \cos 3\psi \\ y &= \frac{3}{4}a \sin \psi - \frac{1}{4}a \sin 3\psi \end{aligned} \right\} \dots \dots \dots (1)$$

Substituting the values of  $\cos 3\psi$  and  $\sin 3\psi$  from the formulas,

$$\cos 3\psi = 4 \cos^3 \psi - 3 \cos \psi,$$

and

$$\sin 3\psi = 3 \sin \psi - 4 \sin^3 \psi,$$

we have

$$\left. \begin{aligned} x &= a \cos^3 \psi \\ y &= a \sin^3 \psi \end{aligned} \right\}; \dots \dots \dots (2)$$

whence  $x^{\frac{1}{3}} = a^{\frac{1}{3}} \cos^3 \psi$ , and  $y^{\frac{1}{3}} = a^{\frac{1}{3}} \sin^3 \psi$ .

Adding, we have  $x^{\frac{1}{3}} + y^{\frac{1}{3}} = a^{\frac{1}{3}}$ ,  $\dots \dots \dots (3)$

the rectangular equation of the curve.

## CHAPTER IX.

### APPLICATIONS OF THE DIFFERENTIAL CALCULUS TO PLANE CURVES.

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#### XXIII.

#### *The Equation of the Tangent.*

**162.** THE equation of the curve being given in the form  $y=f(x)$ , the inclination of the tangent at any point is determined by the equation

$$\tan \phi = \frac{dy}{dx} = f'(x).$$

Hence, if  $(x_1, y_1)$  be a point of the curve, the equation of a tangent at  $(x_1, y_1)$  will be found by giving to the direction-ratio  $m$ , in the general equation

$$y - y_1 = m(x - x_1),$$

the value  $\left. \frac{dy}{dx} \right]_{x_1}$ ; thus

$$y - y_1 = \left. \frac{dy}{dx} \right]_{x_1} (x - x_1), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or 
$$y - y_1 = f'(x_1)(x - x_1). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For example, in the case of the semi-cubical parabola

$$y^3 = ax^2,$$

we have  $\frac{dy}{dx} = \frac{3}{2} \sqrt{\frac{a}{x}}$ .

The point  $(a, a)$  is a point of this curve; the equation of the tangent at this point is, therefore,

$$3y - 2x = a.$$

$$\begin{aligned} u &= \frac{3}{2} \sqrt{\frac{a}{x}} \\ y &= a^{\frac{3}{2}} x^{-\frac{1}{2}} \\ dy &= a^{\frac{3}{2}} \cdot \frac{1}{2} x^{-\frac{3}{2}} \cdot (-1) \\ &= -\frac{1}{2} a^{\frac{3}{2}} x^{-\frac{3}{2}} \end{aligned}$$

### *The Equation of the Normal.*

163. A perpendicular to the tangent at its point of contact is called a *normal* to the curve.

The coordinate axes being rectangular, the direction-ratio of the normal is the negative reciprocal of that of the tangent; for the inclination of the normal is  $\frac{1}{2}\pi + \phi$ , and

$$\tan(\frac{1}{2}\pi + \phi) = -\cot \phi.$$

The equation of the normal may, therefore, be written thus—

$$y - y_1 = -\left[\frac{dx}{dy}\right]_{x_1} (x - x_1), \quad \dots \quad (1)$$

or 
$$y - y_1 = -\frac{1}{f'(x_1)} (x - x_1). \quad \dots \quad (2)$$

As an illustration, let us take the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

whence

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}.$$

The equation of the normal at any point  $(x_1, y_1)$  of the ellipse is, therefore,

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1} (x - x_1).$$

### Subtangents and Subnormals.

**164.** Denoting by  $s$  the length of the arc measured from some fixed point,  $\frac{ds}{dt}$  denotes the velocity of  $P$ , the generating point of the curve; let  $PT$ , equal to  $ds$ , be measured on the tangent at  $P$ , then  $PQ$  and  $QT$  will represent  $dx$  and  $dy$ , and the angle  $TPQ$  will be  $\phi$ ; hence

$$\cos \phi = \frac{dx}{ds}, \quad \sin \phi = \frac{dy}{ds}, \quad (1)$$

$$\text{and} \quad ds = \sqrt{(dx^2 + dy^2)}. \quad (2)$$

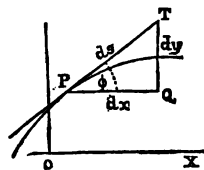


FIG. 33.

**165.** The distance  $PT$  (Fig. 34) on the tangent line intercepted between the point of contact and the axis of  $x$  is sometimes called the *tangent*, and in like manner the intercept  $PN$  is called the *normal*.

From the triangles  $PTR$  and  $NPR$ , we have

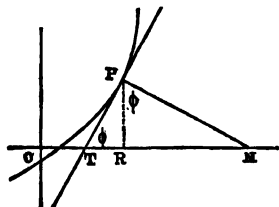


FIG. 34.

$$PT = y \operatorname{cosec} \phi = y \frac{ds}{dy} = y \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]},$$

$$PN = y \sec \phi = y \frac{ds}{dx} = y \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}.$$

The projections of these lines on the axis of  $x$ , that is  $TR$  and  $RN$ , are called the *subtangent* and the *subnormal*.

From the same triangles, we have

the subtangent,  $TR = y \cot \phi = y \frac{dx}{dy}$ ,

and the subnormal,  $RN = y \tan \phi = y \frac{dy}{dx}$ .

*The Perpendicular from the Origin upon the Tangent.*

166. If a perpendicular  $p$  to the tangent  $PR$  be drawn from the origin, we have, from the triangles in Fig.

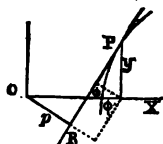


FIG. 35.

35,

$$p = x \sin \phi - y \cos \phi, \quad \dots \quad (1)$$

$\phi - 90^\circ$  being taken as the positive direction of  $p$ . Substituting the values of  $\sin \phi$  and  $\cos \phi$ , equation (1) becomes

$$p = \frac{xdy - ydx}{ds} = \frac{xdy - ydx}{\sqrt{(dx^2 + dy^2)}} \dots \dots \quad (2)$$

For example, let us determine  $p$  in the case of the four-cusped hypocycloid,

$$x = a \cos^3 \psi, \quad y = a \sin^3 \psi.$$

Differentiating,

$$dx = -3a \cos^2 \psi \sin \psi d\psi, \quad \text{and} \quad dy = 3a \sin^2 \psi \cos \psi d\psi;$$

whence  $ds = 3a \sin \psi \cos \psi d\psi$ .

Substituting in equation (2) we obtain

$$p = a \cos^3 \psi \sin \psi + a \sin^3 \psi \cos \psi = a \sin \psi \cos \psi = \sqrt[3]{(axy)}.$$

To ascertain the direction of  $p$  it is necessary to determine

$\phi$ . The ambiguity in the value of  $\phi$  as determined from the equation  $\tan \phi = \frac{dy}{dx}$  may be removed by means of one of the formulas of Art. 164. Thus, in the present case, we have

$$\tan \phi = -\tan \psi, \text{ whence } \phi = -\psi, \text{ or } \phi = \pi - \psi;$$

but, since 
$$\cos \phi = \frac{dx}{ds} = -\cos \psi,$$

we must take 
$$\phi = \pi - \psi.$$

The direction of  $\rho$  when positive is therefore  $\frac{1}{2}\pi - \psi$ .

### Examples XXIII.

1. In the case of the parabola of the  $n$ th degree

$$a^{n-1}y = x^n,$$

find the equations of the tangent and the normal at the point  $(a, a)$ .

2. Find the subtangent and the subnormal of the parabola

$$y^2 = 4ax.$$

3. Prove that the subtangent of the exponential curve

$$y = a^x$$

is constant, and find the ordinate of the point of contact when the tangent passes through the origin.  $\epsilon$ .

4. Find the subnormal of the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

5. Find the subtangent of the curve

$$a^{n-1}y = x^n.$$

6. In the case of the parabola

$$y^2 = 4ax,$$

find  $p$  in terms of  $x$ .

$$\text{For the upper branch, } p = -\frac{x \sqrt{a}}{\sqrt{a+x}}.$$

7. Find, in terms of  $\psi$ , the equation of the tangent to the four-cusped hypocycloid (Art. 161), and thence show that the part intercepted between the axes is of constant length.

8. In the case of the epicycloid, find the value of  $ds$  in terms of the auxiliary angle  $\psi$ . See Art. 159.

$$ds = 2(a+b) \sin \frac{a\psi}{2b} d\psi.$$

9. Determine the value of  $p$  in the case of the epicycloid employing the value of  $ds$  determined in the preceding example.

$$p = (a+2b) \sin \frac{a\psi}{2b}.$$

## XXIV.

### *Polar Coordinates.*

**167.** When the equation of a curve is given in polar co-ordinates the vectorial angle  $\theta$  is usually taken as the independent variable; hence, denoting by  $s$  an arc of the curve, it is usual to assume that  $ds$  and  $d\theta$  have the same sign; that is, that  $\frac{ds}{d\theta}$  is positive.

In Fig. 36 let  $PT$ , a portion of the tangent line, represent  $ds$ ; then, producing  $r$ , let the rectangle  $PT$  be completed, and

let  $\psi$  denote the angle  $TPS$ ; that is, the angle between the *positive directions* of  $r$  and  $s$ . The resolved velocities of  $P$  along and perpendicular to the radius vector are  $\frac{dr}{dt}$  and  $\frac{rd\theta}{dt}$ , the latter being the velocity which  $P$  would have if  $r$  were constant; that is, if  $P$  moved in a circle described with  $r$  as a radius. Hence we have

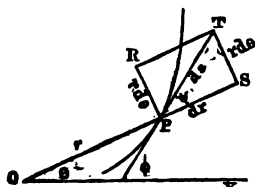


FIG. 36.

$$PS = dr \quad \text{and} \quad PR = rd\theta.$$

From the triangle  $PST$ , we derive

$$\tan \psi = \frac{rd\theta}{dr}, \quad \sin \psi = \frac{rd\theta}{ds}, \quad \cos \psi = \frac{dr}{ds}, \quad \dots (1)$$

and

$$\frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}. \quad \dots (2)$$

**168.** The second of equations (1) shows that, in accordance with the assumption that  $ds$  has the sign of  $d\theta$ , the value of  $\psi$  will always be either in the first or in the second quadrant.

The first of equations (1) is equivalent to

$$\cot \psi = \frac{dr}{rd\theta}, \quad \dots (3)$$

which shows that  $\cot \psi$  is the *logarithmic derivative* of  $r$  regarded as a function of  $\theta$ . Thus in the case of the logarithmic spiral

$$r = ae^{n\theta}$$

we have  $\log r = \log a + n\theta$ ,

hence  $\cot \psi = n$



whence it follows that, in the case of this curve,  $\psi$  is constant. See Art. 154.

**169.** It is frequently convenient to employ in place of the radius vector its reciprocal, which is usually denoted by  $u$ ; then

$$r = \frac{1}{u}, \text{ and } dr = -\frac{du}{u^2}. \quad (4)$$

Making these substitutions in equations (2) and (3) we have

$$\frac{ds}{d\theta} = \frac{1}{u^2} \sqrt{\left[ u^2 + \left( \frac{du}{d\theta} \right)^2 \right]}. \quad (5)$$

and

$$\cot \psi = -\frac{du}{u d\theta}. \quad (6)$$

### *Polar Subtangents and Subnormals.*

**170.** Let a straight line perpendicular to the radius vector be drawn through the pole, and let the tangent and the normal meet this line in  $T$  and  $N$  respectively; then the projections of  $PT$  and  $PN$  upon this line, that is  $OT$  and  $ON$ , are called respectively the *polar subtangent* and the *polar subnormal*. In Fig. 37,  $OPT = \psi$ ; whence

$$OT = r \tan \psi = r^2 \frac{d\theta}{dr} = -\frac{d\theta}{\frac{dr}{r^2}},$$

and

$$ON = r \cot \psi = \frac{dr}{d\theta} = -\frac{du}{u^2 d\theta}.$$

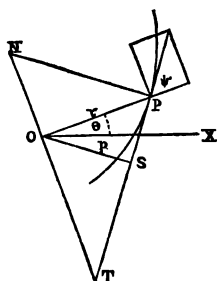


FIG. 37.

Fig. 37 shows that the value of  $OT$  is positive when its

direction is  $\theta - 90^\circ$ ; that of  $ON$  is, on the other hand, positive when its direction is  $\theta + 90^\circ$ .

*The Perpendicular from the Pole upon the Tangent.*

171. Let  $p$  denote the perpendicular distance from the pole to the tangent; then, from Fig. 37 we obtain

$$p = r \sin \psi = r^2 \frac{d\theta}{ds} = \frac{r^2}{\sqrt{\left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right]}}. \quad \dots (1)$$

These expressions give positive values for  $p$ , because  $\frac{ds}{d\theta}$  is assumed to be positive, and Fig. 37 shows that  $p$  has the direction  $\phi - 90^\circ$ ,  $\phi$  being the angle which the positive direction of  $s$  makes with the initial line.

The relation between  $p$  and  $u$  is obtained thus:—from (1) we have

$$\frac{1}{p^2} = \frac{ds^2}{r^4 d\theta^2},$$

and, transforming by the formulas of Art. 169,

$$\frac{1}{p^2} = u^2 + \left( \frac{du}{d\theta} \right)^2. \quad \dots (2)$$

172. The expression deduced below for the function  $u + \frac{d^2 u}{d\theta^2}$  is frequently useful.

Differentiating (2), we have

$$2u du + 2 \frac{du}{d\theta} \cdot \frac{d^2 u}{d\theta^2} = - \frac{2dp}{p^3};$$

hence 
$$\left(u + \frac{d^2u}{d\theta^2}\right) du = -\frac{dp}{p^2},$$

or since 
$$du = -\frac{dr}{r^2},$$

$$u + \frac{d^2u}{d\theta^2} = \frac{r^2}{p^2} \cdot \frac{dp}{dr}.$$

*The Perpendicular upon an Asymptote.*

173. When the point of contact  $P$  passes to infinity the tangent at  $P$  becomes an asymptote, and the subtangent  $OT$  coincides with the perpendicular upon the asymptote. Hence ( $\theta_1$  denoting a value of  $\theta$  for which  $r$  is infinite) the length of this perpendicular is given by the expression  $-\left[\frac{d\theta}{du}\right]_{\theta_1}$ , and like the polar subtangent is, when positive, to be laid off in the direction  $\theta_1 - 90^\circ$ .

This expression for the perpendicular upon the asymptote is also easily derived by evaluating that given in Art. 146. Thus—

$$r \sin(\theta_1 - \theta) \Big]_{\theta_1} = \frac{\sin(\theta_1 - \theta)}{u} \Big]_{\theta_1} = -\left[\frac{d\theta}{du}\right]_{\theta_1}.$$

*Points of Inflexion.*

174. When, as in Fig. 37, the curve lies between the tangent and the pole, it is obvious that  $r$  and  $p$  will increase and decrease together; that is,  $\frac{dp}{dr}$  will be positive. When on the other hand the curve lies on the other side of the tangent,  $\frac{dp}{dr}$  is negative. Hence at a point of inflexion  $\frac{dp}{dr}$  must change sign.

Now, since  $p$  is always positive, it follows from the equation deduced in Art. 172 that the sign of this expression is the same as that of

$$u + \frac{d^2u}{d\theta^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

hence *at a point of inflexion this expression must change sign.*

**175.** As an illustration, let us determine the point of inflexion of the curve traced in Art. 147; viz.,

$$r = \frac{a\theta^3}{\theta^3 - 1}.$$

In this case 
$$u = \frac{1}{a} (1 - \theta^{-3});$$

whence 
$$\frac{du}{d\theta} = \frac{2}{a} \theta^{-3}, \quad \text{and} \quad \frac{d^2u}{d\theta^2} = -\frac{6}{a} \theta^{-4};$$

therefore 
$$u + \frac{d^2u}{d\theta^2} = \frac{1}{a} (1 - \theta^{-3} - 6\theta^{-4})$$
$$= \frac{\theta^4 - \theta^3 - 6}{a\theta^4}.$$

Putting this expression equal to zero, the real roots are

$$\theta = \pm \sqrt[4]{3},$$

and it is evident that, as  $\theta$  passes through either of these values, the expression  $u + \frac{d^2u}{d\theta^2}$  changes sign. Hence the points of inflexion are determined by

$$\theta = \pm \sqrt[4]{3} \quad \text{and} \quad r = \frac{3a}{2}.$$

**Examples XXIV.**

1. Prove that, in the case of the lemniscata  $r^2 = a^2 \cos 2\theta$ ,

$$\psi = 2\theta + \frac{1}{2}\pi, \quad \text{and} \quad \frac{ds}{d\theta} = \frac{a^2}{r}.$$

2. Find the subtangent of the lituus  $r^3 = \frac{a^3}{\theta}$ , and prove that the perpendicular from the origin upon the tangent is

$$\frac{2a \sqrt{\theta}}{\sqrt{(1 + 4\theta^3)}}.$$

3. Find the polar subtangent of the spiral  $r(\epsilon^\theta + \epsilon^{-\theta}) = a$ .

$$- \frac{a}{\epsilon^\theta - \epsilon^{-\theta}}.$$

4. Find the value of  $p$  in the case of the curve  $r^n = a^n \sin n\theta$ .

$$p = a (\sin n\theta)^{1 + \frac{1}{n}}.$$

5. In the case of the parabola referred to the focus

$$r = \frac{2a}{1 + \cos \theta}, \quad \text{prove that } p^2 = ar.$$

6. In the case of the equilateral hyperbola

$$r^2 \cos 2\theta = a^2, \quad \text{prove that } p = \frac{a^2}{r}.$$

7. In the case of the lemniscata

$$r^2 = a^2 \cos 2\theta, \quad \text{prove that } p = \frac{r^3}{a^2}.$$

8. In the case of the ellipse  $r = \frac{a(1 - e^2)}{1 - e \cos \theta}$ , the pole being at

the focus, determine  $p$ .

$$p = \frac{a(1 - e^2)}{\sqrt{(1 - 2e \cos \theta + e^2)}}.$$

9. In the case of the cardioid

$$r = a(1 + \cos \theta), \text{ prove that } r^3 = 2ap^3.$$

10. Show that the curve  $r\theta \sin \theta = a$  has a point of inflexion at which  $r = \frac{2a}{\pi}$ .

## XXV.

### *Curvature.*

176. If, while a point  $P$  moves along a given curve at the rate  $\frac{ds}{dt}$ , it be regarded as carrying with it the tangent and normal lines, each of these lines will rotate about the moving point  $P$  at the angular rate  $\frac{d\phi}{dt}$ ,  $\phi$  denoting the inclination of the tangent line to the axis of  $x$ .

The point  $P$  is always moving in a direction perpendicular to the normal with the velocity  $\frac{ds}{dt}$ . Let us consider the

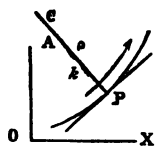


FIG. 38.

motion of a point  $A$  on the normal at a given distance  $k$  from  $P$  on the concave side of the arc. While this point is carried forward by the motion of  $P$  with the velocity  $\frac{ds}{dt}$  in a direction perpendicular to the normal, it is at the same time carried backward, by the rotation of this line about  $P$ , with the

velocity  $\frac{k d\phi}{dt}$ ; since this is the velocity with which  $A$  would move if the point  $P$  occupied a fixed position in the plane; and the *direction* of this motion is evidently directly opposite to that of  $P$ . Hence the actual velocity of  $A$  will be

$$\frac{ds}{dt} - k \frac{d\phi}{dt},$$

in a direction parallel to the tangent at  $P$ .

Let  $\rho$  denote the value of  $k$  which reduces this expression to zero, and let  $C$  (Fig. 38) be the corresponding position of  $A$ : then,

$$\frac{ds}{dt} - \rho \frac{d\phi}{dt} = 0;$$

whence 
$$PC = \rho = \frac{ds}{d\phi}. \quad \dots \dots \dots (1)$$

177. The value of  $\rho$  determined by this equation is, in general, variable; for, if the point  $P$  move along the curve with a given linear velocity  $\frac{ds}{dt}$ , the angular velocity  $\frac{d\phi}{dt}$  will generally be variable. If however we suppose the angular velocity  $\frac{d\phi}{dt}$  to become constant, at the instant when  $P$  passes a given position on the curve,  $\frac{ds}{d\phi}$ , the value of  $\rho$ , will likewise become constant, and  $C$  will remain stationary. When this hypothesis is made, the curvature of the path of  $P$  becomes constant, for  $P$  describes a circle whose centre is  $C$ , and whose radius is  $\rho$ . Hence this circle is called *the circle of curvature* corresponding to the given position of  $P$ ;  $C$  is accordingly called *the centre of curvature*, and  $\rho$  is called *the radius of curvature*.

### *The Direction of the Radius of Curvature.*

178. If in Fig. 38 the arrow indicates the positive direction of  $s$ ; the case represented is that in which  $\phi$  and  $s$  increase together, and therefore the value of  $\rho$  as determined by equation (1), Art. 176, is positive. Hence it is evident that when  $\rho$  is positive its direction from  $P$  is that of  $PC$  in Fig. 38; namely,  $\phi + 90^\circ$ . In other words, to a person looking along the curve in the positive direction of  $ds$ ,  $\rho$ , when positive, is laid off on *the left-hand side of the curve*.

For example, let the curve be the four-cusped hypocycloid,

$$x = a \cos^3 \psi, \quad y = a \sin^3 \psi.$$

It was shown in Art. 166 that for this curve

$$ds = 3a \sin \psi \cos \psi d\psi, \quad \text{and} \quad \phi = \pi - \psi;$$

hence 
$$d\phi = -d\psi,$$

and 
$$\rho = \frac{ds}{d\phi} = -3a \sin \psi \cos \psi. \quad \dots \dots (1)$$

When  $\psi$  is in the first quadrant  $\rho$  is negative; its direction is therefore  $\phi - \frac{1}{2}\pi = \frac{1}{2}\pi - \psi$ , which is in the first quadrant. When  $\psi$  is in the second quadrant  $\rho$  is positive and its direction is  $\phi + \frac{1}{2}\pi = \frac{3}{2}\pi - \psi$ , which is in the second quadrant.

### *The Radius of Curvature in Rectangular Coordinates.*

179. To express  $\rho$  in terms of derivatives with reference to  $x$ , we have



$$\phi = \tan^{-1} \frac{dy}{dx}, \text{ and } \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

hence

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2},$$

and

$$\rho = \frac{\frac{ds}{dx}}{\frac{d\phi}{dx}} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \quad \dots \dots (1)$$

Since  $\frac{ds}{dx}$  is assumed to be positive,  $\phi$  should be so taken as to cause  $x$  to increase with  $s$ , and it must be remembered that the direction of  $\rho$  is  $\phi + 90^\circ$  when  $\rho$  is positive, in accordance with the remark in the preceding article.

**180.** To illustrate the application of the above formula, we find the radius of curvature of the ellipse

$$y = \pm \frac{b}{a} \sqrt{(a^2 - x^2)}. \quad \dots \dots (1)$$

Differentiating,

$$\frac{dy}{dx} = \mp \frac{bx}{a\sqrt{(a^2 - x^2)}}, \quad \dots \dots (2)$$

and

$$\frac{d^2y}{dx^2} = \mp \frac{ab}{(a^2 - x^2)^{\frac{3}{2}}}. \quad \dots \dots (3)$$

Putting  $b = a\sqrt{(1 - e^2)}$  we obtain

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{a^2 - e^2x^2}{a^2 - x^2};$$

whence, substituting in equation (1) of the preceding article,

$$\rho = \mp \frac{(a^2 - e^2 x^2)^{\frac{3}{2}}}{a^2 \sqrt{1 - e^2}} \cdot \cdot \cdot \cdot \cdot (4)$$

*Expressions for  $\rho$  in which  $x$  is not the Independent Variable.*

181. To express  $\rho$  in terms of derivatives with reference to  $y$ , we have

$$\phi = \cot^{-1} \frac{dx}{dy}, \quad \text{and} \quad \frac{ds}{dy} = \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]};$$

$$\text{whence} \quad \frac{d\phi}{dy} = - \frac{\frac{d^2x}{dy^2}}{1 + \left(\frac{dx}{dy}\right)^2}, \quad \text{and} \quad \rho = - \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}.$$

In this case  $ds$  and  $dy$  were assumed to have the same sign, hence  $\phi$  must be taken so as to cause  $y$  to increase.

182. When  $x$  and  $y$  are expressed in terms of a third variable we employ the formula deduced below.

Differentiating

$$\phi = \tan^{-1} \frac{dy}{dx},$$

both  $dx$  and  $dy$  being regarded as variable, we have

$$d\phi = \frac{\frac{dx \, d^2y - dy \, d^2x}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} = \frac{dx \, d^2y - dy \, d^2x}{dx^2 + dy^2};$$

$$\text{whence} \quad \rho = \frac{ds}{d\phi} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx \, d^2y - dy \, d^2x} \cdot \cdot \cdot \cdot \cdot (1)$$

## Examples XXV.

1. Find the radius of curvature of the
- cycloid*

$$x = a(\psi - \sin \psi), \quad y = a(1 - \cos \psi).$$

Prove that  $\phi = \frac{1}{2}(\pi - \psi)$ , and use  $\rho = \frac{ds}{d\phi}$ .

$$\rho = -2\sqrt{2ay}.$$

2. Find the radius of curvature of the
- parabola*
- $y^2 = 4ax$
- .

$$\rho = \mp 2 \frac{(a+x)^{\frac{3}{2}}}{\sqrt{a}}.$$

3. Find the radius of curvature of the
- catenary*

$$y = \frac{c}{2} \left( e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right),$$

and show that its numerical value equals that of the normal at the same point. See Art. 165.

$$\rho = \frac{y^2}{c}.$$

4. Find the radius of curvature of the
- semi-cubical parabola*

$$ay^2 = x^3.$$

$$\rho = \frac{(4a + 9x)^{\frac{3}{2}} x^{\frac{1}{2}}}{6a}.$$

5. Find the radius of curvature of the
- logarithmic curve*

$$y = ae^{\frac{x}{c}}.$$

$$\rho = \frac{[c^2 + y^2]^{\frac{3}{2}}}{cy}.$$

6. Find the radius of curvature of the *cissoid*

$$y = \frac{x^{\frac{3}{2}}}{(2a - x)^{\frac{1}{2}}}.$$

$$\rho = \frac{a \sqrt{x} (8a - 3x)^{\frac{3}{2}}}{3 (2a - x)^{\frac{3}{2}}}.$$

7. Find the radius of curvature of the *parabola*

$$\sqrt{x} + \sqrt{y} = 2\sqrt{a}.$$

$$\rho = \frac{(x + y)^{\frac{3}{2}}}{\sqrt{a}}.$$

8. Find the radius of curvature of the *cubical parabola*

$$a^2 y = x^3.$$

$$\rho = \frac{(a^4 + 9x^4)^{\frac{3}{2}}}{6a^2 x}.$$

9. Find the radius of curvature of the *prolate cycloid*

$$x = a\psi - b \sin \psi, \quad y = a - b \cos \psi.$$

$$\rho = \frac{(a^2 + b^2 - 2ab \cos \psi)^{\frac{3}{2}}}{b(a \cos \psi - b)}.$$

## XXVI.

### *Envelopes.*

**183.** The curves determined by an equation involving  $x$  and  $y$  together with constants to which arbitrary values may be assigned are said to constitute a *system* of curves. The arbitrary constants are called *parameters*. When but one of

the parameters is regarded as variable, denoting it by  $\alpha$ , the general equation of the system of curves may be expressed thus :

$$f(x, y, \alpha) = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the curves of a system mutually intersect (the intersections not being fixed points), there usually exists a curve which touches each curve of the system obtained by causing the value of  $\alpha$  to vary.

For example, the ellipses whose axes are fixed in position, and whose semi-axes have a constant sum, constitute such a system ; and, if we regard the ellipse as varying continuously from the position in which one semi-axis is zero to that in which the other is zero, it is evident that the boundary of that portion of the plane which is swept over by the perimeter of the varying ellipse is a curve to which the ellipse is tangent in all its positions. A curve having this relation to a given system of curves is called the *envelope of the system*.

Every point on an envelope may be regarded as the limiting position of the point of intersection of two members of the given system of curves, when the difference between the corresponding values of  $\alpha$  is indefinitely diminished. For this reason, the envelope is sometimes called *the locus of the ultimate intersections* of the curves of the given system.

184. If we differentiate equation (1) of the preceding article (regarding  $\alpha$  as a variable as well as  $x$  and  $y$ ) the resulting equation will be of the general form

$$f'_x(x, y, \alpha) dx + f'_y(x, y, \alpha) dy + f'_\alpha(x, y, \alpha) d\alpha = 0. \quad (2)$$

In this equation each term may be separately obtained by differentiating the given equation on the supposition that the quantity indicated by the subscript is alone variable. See Art. 64.

From equation (2) we derive

$$\frac{dy}{dx} = -\frac{f'_x(x, y, \alpha)}{f'_y(x, y, \alpha)} - \frac{f'_\alpha(x, y, \alpha) d\alpha}{f'_y(x, y, \alpha) dx} \cdot \cdot \cdot \cdot (3)$$

In Fig. 39 let  $PC$  be the curve corresponding to a particular value of  $\alpha$ , and let  $P$  be the point  $(x, y)$ ; then the expression for  $\frac{dy}{dx}$  given in equation (3) determines the direction in which the point  $P$  is actually moving when  $x, y$ , and  $\alpha$  vary simultaneously. This direction depends there-

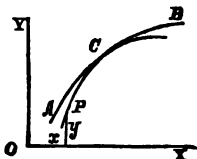


FIG. 39.

fore in part upon the arbitrary value given to the ratio  $\frac{d\alpha}{dx}$ .

185. Now if  $\alpha$  were constant  $d\alpha$  would vanish, and equation (3) would become

$$\frac{dy}{dx} = -\frac{f'_x(x, y, \alpha)}{f'_y(x, y, \alpha)} \cdot \cdot \cdot \cdot (4)$$

This expression for  $\frac{dy}{dx}$  determines the direction in which  $P$  moves when  $PC$  is a fixed curve.

Let  $AB$  be an arc of the envelope, and let  $C$  be its point of contact with  $PC$ . Now, if  $P$  be placed at the point  $C$ , it is obvious that it can move only in the direction of the common tangent at  $C$ , *whether  $\alpha$  be fixed or variable*. It follows therefore that, at every point at which a curve belonging to the system touches the envelope, the expressions for  $\frac{dy}{dx}$  given in equations (3) and (4) must be identical in value.

Assuming that  $f'_x(x, y, \alpha)$  and  $f'_y(x, y, \alpha)$  do not become infinite for any finite values of  $x$  and  $y$ , the above condition requires that

$$f'_\alpha(x, y, \alpha) = 0 \cdot \cdot \cdot \cdot (5)$$

Hence the coordinates of every point of the envelope must satisfy simultaneously equations (5) and (1); the equation of the envelope is therefore obtained by eliminating  $\alpha$  between these two equations.

**186.** Let it be required to find the *envelope of the circles having for diameters the double ordinates of the parabola*

$$y^2 = 4ax.$$

If we denote by  $\alpha$  the abscissa of the centre of the variable circle, its radius will be the ordinate of the point on the parabola of which  $\alpha$  is the abscissa, the equation of the circle will therefore be

$$y^2 + (x - \alpha)^2 - 4a\alpha = 0. \quad \dots \dots (1)$$

Differentiating with reference to the variable parameter  $\alpha$ , we have

$$-2(x - \alpha) - 4a = 0,$$

or

$$\alpha = 2a + x; \dots \dots (2)$$

substituting in (1), and reducing, we obtain

$$y^2 = 4a(a + x). \dots \dots (3)$$

The envelope is, therefore, a parabola equal to the given parabola and having its focus at the vertex of the given parabola.

### *Two Variable Parameters.*

**187.** When the equation of the given curve contains two variable parameters connected by an equation, only one of these parameters can be regarded as arbitrary, since, by means of the equation connecting them, one of the parameters can be eliminated. Instead, however, of eliminating one of the parameters at once, it is often better to proceed as in the following example.

Required, *the envelope of a straight line of fixed length  $a$ , which moves with its extremities on two rectangular axes.*

Denoting the intercepts on the axes by  $\alpha$  and  $\beta$ , the equation of the line is

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1, \dots \dots \dots (1)$$

$\alpha$  and  $\beta$  being two variable parameters which, by the conditions of the problem, are connected by the relation

$$\alpha^2 + \beta^2 = a^2. \dots \dots \dots (2)$$

Differentiating (1) and (2) with respect to  $\alpha$  and  $\beta$  as variables, we have

$$\frac{x d\alpha}{\alpha^2} + \frac{y d\beta}{\beta^2} = 0, \dots \dots \dots (3)$$

and 
$$\alpha d\alpha + \beta d\beta = 0. \dots \dots \dots (4)$$

We have now four equations from which we are to eliminate  $\alpha$ ,  $\beta$ , and the ratio  $\frac{d\alpha}{d\beta}$ . Transposing and dividing (3) by (4), we obtain

$$\frac{x}{\alpha^2} = \frac{y}{\beta^2}.$$

Substituting in (1) the value of  $y$  derived from the last equation, we have

$$x(\alpha^2 + \beta^2) = \alpha^2;$$

whence by equation (2)

$$\alpha = x^{\frac{1}{2}} a^{\frac{1}{2}}.$$



In like manner we find

$$\beta = y^{\frac{1}{2}} a^{\frac{1}{2}}.$$

Hence, substituting in (2)

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$$

The envelope is therefore a *four-cusped hypocycloid*.

### *Evolutes.*

188. In Fig. 40 let  $C$  be the centre of curvature of the given curve: this point is so determined (see Art. 176) as to have no motion in a direction perpendicular to the normal  $PC$ , but since  $\rho$  is in general variable, it has a motion in the direction  $PC$ . Hence  $C$  describes a curve to which the normal  $PC$  is always tangent at the point  $C$ . Moreover, since  $P$  has no motion in the direction  $PC$ , if we regard  $P$  as a fixed point on this line, the rate of  $C$  along this moving line will be identical with its rate along the curve which it describes. Hence the motion of  $PC$  is the same as that of a tangent line rolling upon the curve described by  $C$ , while  $P$ , a fixed point of this tangent, describes the original curve.

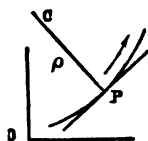


FIG. 40.

The curve described by the centre of curvature  $C$  is therefore called the *evolute* of the curve described by  $P$ , and the latter is called an *involute* of the former.

189. Since the evolute of a given curve is the curve to which all the normals to the given curve are tangent, it is evidently the envelope of these normals.

The equation of the normal at the point  $(x, y)$  of a given curve may be written in the form

$$x' - x + (y' - y) \frac{dy}{dx} = 0, \quad . \quad . \quad . \quad . \quad (1)$$

$(x', y')$  being any point of the normal. See Art. 163.

In this equation  $y$  and  $\frac{dy}{dx}$  are functions of  $x$  determined by the equation of the given curve, and  $x$  is to be regarded as the arbitrary parameter. Hence, differentiating with reference to  $x$ , we have

$$-1 - \left(\frac{dy}{dx}\right)^2 + (y' - y) \frac{d^2y}{dx^2} = 0. \quad . \quad . \quad . \quad . \quad (2)$$

The equation of the evolute is therefore the relation between  $x'$  and  $y'$  which arises from the elimination of  $x$  between equations (1) and (2).

**190.** As an illustration, let it be required to find the evolute of the common parabola

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}};$$

whence  $\frac{dy}{dx} = \left(\frac{a}{x}\right)^{\frac{1}{2}},$  and  $\frac{d^2y}{dx^2} = -\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}.$

Substituting, we obtain from equation (2) of the preceding article

$$x^{\frac{3}{2}} = -\frac{1}{2}a^{\frac{1}{2}}y';$$

whence, from equation (1) of the same article,

$$27ay^2 = 4(x' - 2a)^3,$$

the equation of the evolute, which is, therefore, a *semi-cubical parabola* having its cusp at the point  $(2a, 0)$ .

191. It is frequently desirable to express the equation of the normal in terms of some parameter other than  $x$  before differentiating. Thus, let us determine *the evolute of the ellipse* by means of the equation of the normal in terms of the eccentric angle.

The equations of the ellipse are

$$x = a \cos \psi, \quad \text{and} \quad y = b \sin \psi;$$

$$\text{whence} \quad dx = -a \sin \psi d\psi, \quad \text{and} \quad dy = b \cos \psi d\psi.$$

Substitution in the equation of the normal,

$$(x' - x) dx + (y' - y) dy = 0,$$

$$\text{gives} \quad ax' \sin \psi - by' \cos \psi - (a^2 - b^2) \sin \psi \cos \psi = 0.$$

Differentiating, we have

$$ax' \cos \psi + by' \sin \psi - (a^2 - b^2) (\cos^2 \psi - \sin^2 \psi) = 0;$$

eliminating  $y'$  and  $x'$  successively, and dropping the accents,

$$ax = (a^2 - b^2) \cos^3 \psi \quad \text{and} \quad by = -(a^2 - b^2) \sin^3 \psi;$$

$$\text{whence} \quad (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

### Examples XXVI.

1. Find the envelope of the system of parabolas represented by the equation

$$y^2 = \frac{\alpha^2}{c} (x - \alpha),$$

in which  $\alpha$  is an arbitrary parameter and  $c$  a fixed constant.

$$y^2 = \frac{4}{27c} x^3.$$

2. Find the envelope of the circles described on the double ordinates of an ellipse as diameters.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

3. Find the envelope of the ellipses, the product of whose semi-axes is equal to the constant  $c^2$ .

$$\text{The conjugate hyperbolas, } 2xy = \pm c^2.$$

4. Find the envelope of a perpendicular to the normal to the parabola,  $y^2 = 4ax$ , drawn through the intersection of the normal with the axis.

$$y^2 = 4a(2a - x).$$

5. Find the envelope of the ellipses whose axes are fixed in position, and whose semi-axes have a constant sum  $c$ .

$$\text{The four-cusped hypocycloid, } x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

6. Given the equation of the *catenary*

$$y = \frac{a}{2} \left( e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right);$$

prove that

$$y' = 2y, \quad \text{and} \quad x' = x - \frac{y}{2} \left( e^{\frac{x}{a}} - e^{-\frac{x}{a}} \right),$$

and deduce the equation of the evolute.

$$x' = a \log \frac{y' \pm (y'^2 - 4a^2)^{\frac{1}{2}}}{2a} \mp \frac{y'}{4a} (y'^2 - 4a^2)^{\frac{1}{2}}.$$

7. Derive the equation of the evolute to the hyperbola, its equations in terms of an auxiliary angle being

$$x = a \sec \psi \quad \text{and} \quad y = b \tan \psi.$$

The equation of the normal is

$$ax \sin \psi + by = (a^2 + b^2) \tan \psi,$$

and the equation of the evolute is

$$a^{\frac{2}{3}}x^{\frac{2}{3}} - b^{\frac{2}{3}}y^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

8. Find the equation of the evolute of the cycloid.

The equation of the normal is

$$x + y \frac{\sin \psi}{1 - \cos \psi} - a\psi = 0.$$

The equations of the evolute are

$$x = a(\psi + \sin \psi) \quad \text{and} \quad y = -a(1 - \cos \psi).$$

The evolute is therefore a cycloid situated below the axis of  $x$ , having its vertex at the origin. See equations (3), Art. 158.

## CHAPTER X.

### FUNCTIONS OF TWO OR MORE VARIABLES

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#### XXVII.

#### *The Derivative Regarded as the Limit of a Ratio.*

**192.** THE difference between two values of a variable is frequently expressed by prefixing the symbol  $\Delta$  to the symbol denoting the variable, and the difference between corresponding values of any function of the variable, by prefixing  $\Delta$  to the symbol denoting the function. Hence  $x$  and  $x + \Delta x$  denote two values of the independent variable, and  $\Delta f(x)$  denotes the difference between the corresponding values of  $f(x)$ ; that is, if  $y = f(x)$ ,

$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x). \quad . \quad . \quad . \quad (1)$$

If we put  $\Delta x = 0$ , we shall have  $\Delta y = 0$ ;

hence the ratio 
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad . \quad . \quad . \quad . \quad (2)$$

takes the indeterminate form  $\frac{0}{0}$  when  $\Delta x = 0$ . The value assumed in this case is called *the limiting value of the ratio of the increments*,  $\Delta y$  and  $\Delta x$ , when the absolute values of these increments are diminished indefinitely.

**193.** To determine this limiting value, for a particular value  $a$  of  $x$ , we put  $a$  for  $x$  and  $z$  for  $\Delta x$  in the second member of

equation (2), and evaluate for  $z = 0$ , by the ordinary process (see Art. 82). Thus

$$\left[ \frac{f(a+z) - f(a)}{z} \right]_0 = f'(a). \quad . \quad . \quad . \quad . \quad (3)$$

Therefore when  $\Delta x$  is diminished indefinitely, the limiting value of  $\frac{\Delta y}{\Delta x}$  corresponding to  $x = a$  is  $\left[ \frac{dy}{dx} \right]_a$ , and, since  $a$  denotes any value of  $x$ , we have in general

$$\text{limit of } \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

If we denote by  $e$  the difference between the values of  $\frac{\Delta y}{\Delta x}$  and  $\frac{dy}{dx}$ , we shall have

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + e, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the result established in the preceding article may be expressed thus—

$$e = 0 \quad \text{when} \quad \Delta x = 0;$$

in other words,  $e$  is a quantity that vanishes with  $\Delta x$ .

### *Partial Derivatives.*

194. Let  $u = f(x, y)$ ,

in which  $x$  and  $y$  are two independent variables. The derivative of  $u$  with reference to  $x$ ,  $y$  being regarded as constant, is denoted by  $\frac{d}{dx}u$ , and the derivative of  $u$  with reference to  $y$ ,  $x$  being constant, by  $\frac{d}{dy}u$ . These derivatives are called the *partial* derivatives of  $u$  with reference to  $x$  and  $y$  respectively.





were  $x$  the only variable; and, since  $\frac{d}{dx} u \cdot dx$  is the value which  $du$  assumes when this supposition is made, if we put

$$\frac{d}{dx} u \cdot dx = d_x u,$$

this rate will be denoted by  $\frac{d_x u}{dt}$ . Hence by equation (4), Art. 193, equation (2) becomes

$$\frac{du}{dt} + e = \frac{d_x u}{dt} + e' + \frac{d_y u'}{dt} + e'',$$

in which  $e$ ,  $e'$ , and  $e''$  vanish with  $\Delta t$ ; but when  $\Delta t = 0$ ,  $\Delta x = 0$ , and therefore  $u' = u$ ; hence, putting  $\Delta t = 0$ , we have

$$\frac{du}{dt} = \frac{d_x u}{dt} + \frac{d_y u}{dt}.$$

Therefore

$$du = d_x u + d_y u;$$

that is,

$$du = \frac{d}{dx} u \cdot dx + \frac{d}{dy} u \cdot dy.$$

196. This result is usually written in the form

$$du = \frac{du}{dx} dx + \frac{du}{dy} dy,$$

but when written in this form it must be remembered that the fractions in the second member represent *partial derivatives*, the symbol  $du$  in the numerators standing for the quantities denoted above by  $d_x u$  and  $d_y u$ , which are sometimes called *partial differentials*. The  $du$  that appears in the first member is called the *total differential* of  $u$  when  $x$  and  $y$  are both variable.

The above result is easily extended to functions of more than two independent variables.

**Examples XXVII.**

1. Given
- $u = (x^3 + y^3)^{\frac{1}{3}}$
- , prove that

$$x \frac{du}{dx} + y \frac{du}{dy} = \frac{2}{3}u.$$

2. Given
- $u = \frac{xy}{x+y}$
- , prove that

$$x \frac{du}{dx} + y \frac{du}{dy} = u.$$

3. Given
- $u = \tan^{-1} \left( \frac{x-y}{x+y} \right)^{\frac{1}{2}}$
- , prove that

$$x \frac{du}{dx} + y \frac{du}{dy} = 0.$$

4. Given
- $u = \log_e x$
- , to find
- $\frac{du}{dx}$
- and
- $\frac{du}{dy}$
- .

$$\frac{du}{dx} = \frac{1}{x \log y}; \quad \frac{du}{dy} = \frac{-\log x}{y(\log y)^2}.$$

**XXVIII.***The Second and Higher Derivatives regarded as Limits.*

197. In Art. 193 it is shown that

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + e.$$

In this equation  $e$  is a function of  $x$  and likewise of  $\Delta x$ ; hence the derivative  $\frac{de}{dx}$  is in general a function of  $x$  and of  $\Delta x$ . It is

also proved in the same article that  $e$  becomes zero when  $\Delta x$  vanishes; that is,  $e$  assumes a constant value independent of the value of  $x$  when  $\Delta x$  becomes zero; hence, when  $\Delta x$  is zero, the derivative of  $e$  with reference to  $x$  must take the value zero, whatever be the value of  $x$ ; in other words,

$$\frac{de}{dx} \text{ vanishes with } \Delta x.$$

In a similar manner it may be shown that each of the higher derivatives of  $e$  with reference to  $x$  vanishes when  $\Delta x = 0$ .

198. Since  $\frac{\Delta y}{\Delta x}$  is a function of  $x$ ,  $\Delta \frac{\Delta y}{\Delta x}$  will denote the increment of this function corresponding to  $\Delta x$ . Employing the symbol  $\frac{\Delta}{\Delta x}$  to denote the operation of taking this increment, and dividing the result by  $\Delta x$ , we obtain, by applying to this function the principle expressed in equation (4), Art. 193,

$$\begin{aligned} \frac{\Delta}{\Delta x} \cdot \frac{\Delta y}{\Delta x} &= \frac{d}{dx} \cdot \frac{\Delta y}{\Delta x} + e', \dots \dots \dots (1) \\ &= \frac{d}{dx} \left( \frac{dy}{dx} + e \right) + e' \\ &= \frac{d^2 y}{dx^2} + \frac{de}{dx} + e'. \end{aligned}$$

In this equation both  $e'$  and  $\frac{de}{dx}$  vanish with  $\Delta x$  by the preceding article; hence the sum of these quantities likewise vanishes with  $\Delta x$ , and may be denoted by  $e$ . Thus we write

$$\frac{\Delta}{\Delta x} \cdot \frac{\Delta y}{\Delta x} = \frac{d^2 y}{dx^2} + e. \dots \dots \dots (2)$$

**199.** Since  $\Delta x$  is an arbitrary quantity it may be regarded as constant, whence  $\Delta \frac{\Delta y}{\Delta x}$  is the increment of a fraction whose denominator is constant; but this is evidently equivalent to the result obtained by dividing the increment of the numerator by the denominator; that is,

$$\Delta \frac{\Delta y}{\Delta x} = \frac{\Delta \cdot \Delta y}{\Delta x}.$$

The numerator  $\Delta \cdot \Delta y$  is usually denoted by the symbol  $\Delta^2 y$ ; hence equation (2) may be written thus:

$$\frac{\Delta^2 y}{\Delta x^2} = \frac{d^2 y}{dx^2} + \epsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, since  $\epsilon$  vanishes with  $\Delta x$ , it follows that the second derivative is the limit of the expression in the first member of equation (3).

In a similar manner it may be shown that each of the higher derivatives is the limit of the expression obtained by substituting  $\Delta$  for  $d$  in the symbol denoting the derivative.

### *Higher Partial Derivatives.*

**200.** The partial derivatives of  $u$  with reference to  $x$  and  $y$  are themselves functions of  $x$  and  $y$ . Their partial derivatives, viz.,

$$\frac{d}{dx} \cdot \frac{du}{dx}, \quad \frac{d}{dy} \cdot \frac{du}{dx}, \quad \frac{d}{dx} \cdot \frac{du}{dy}, \quad \text{and} \quad \frac{d}{dy} \cdot \frac{du}{dy},$$

are called partial derivatives of  $u$  of the *second order*.

It will now be shown that the second and third of these derivatives, although results of different operations, are in fact identical; that is, that

$$\frac{d}{dy} \cdot \frac{du}{dx} = \frac{d}{dx} \cdot \frac{du}{dy}.$$

Employing the notation introduced in Art. 195, we have

$$\Delta_x u = f(x + \Delta x, y) - f(x, y);$$

if in this equation we replace  $y$  by  $y + \Delta y$ , we obtain a new value of  $\Delta_x u$ , and, denoting this value by  $\Delta'_x u$ , we have

$$\Delta'_x u = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y).$$

Since this change in the value of  $\Delta_x u$  results from the increment received by  $y$ , the expression for the increment received by  $\Delta_x u$  will be  $\Delta_y (\Delta_x u)$ ; hence

$$\Delta_y (\Delta_x u) = \Delta'_x u - \Delta_x u,$$

or

$$\Delta_y (\Delta_x u) = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) - f(x + \Delta x, y) + f(x, y).$$

The value of  $\Delta_x (\Delta_y u)$ , obtained in a precisely similar manner, is identical with that just given; hence

$$\Delta_y (\Delta_x u) = \Delta_x (\Delta_y u). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since  $\Delta x$  is constant, we have, as in Art. 199,

$$\frac{\Delta_y (\Delta_x u)}{\Delta x} = \Delta_y \cdot \frac{\Delta_x u}{\Delta x}.$$

Hence, dividing both members of equation (1) by  $\Delta x \cdot \Delta y$ , we have

$$\frac{\Delta_y}{\Delta y} \cdot \frac{\Delta_x u}{\Delta x} = \frac{\Delta_x}{\Delta x} \cdot \frac{\Delta_y u}{\Delta y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, employing the symbol  $\frac{\Delta}{\Delta x}$  as in Art. 198,

$$\frac{\Delta}{\Delta y} \cdot \frac{\Delta}{\Delta x} u = \frac{\Delta}{\Delta x} \cdot \frac{\Delta}{\Delta y} u.$$

From this result, by a course of reasoning similar to that employed in Art. 198, we obtain

$$\frac{d}{dy} \cdot \frac{du}{dx} = \frac{d}{dx} \cdot \frac{du}{dy} \cdot \dots \dots \dots (3)$$

**201.** The partial derivatives of the second order are usually denoted by

$$\frac{d^2u}{dx^2}, \quad \frac{d^2u}{dx dy}, \quad \frac{d^2u}{dy^2},$$

the factors  $dx$  and  $dy$  in the denominator of the second being, by virtue of formula (3), interchangeable, as in the case of an ordinary product.

The numerators of the above fractions are of course not identical. Compare Art. 196.

Formula (3) of the preceding article is readily verified in any particular case. Thus, given

$$u = y^x,$$

$$\text{whence} \quad \frac{du}{dx} = y^x \log y, \quad \text{and} \quad \frac{du}{dy} = xy^{x-1};$$

$$\therefore \quad \frac{d}{dy} \cdot \frac{du}{dx} = y^{x-1} (x \log y + 1) = \frac{d}{dx} \cdot \frac{du}{dy}.$$

### Examples XXVIII.

1. Given  $u = \sec(y + ax) + \tan(y - ax)$ , prove that

$$\frac{d^2u}{dx^2} = a^2 \frac{d^2u}{dy^2}.$$

2. Verify the theorem  $\frac{d^2u}{dx dy} = \frac{d^2u}{dy dx}$  when  $u = \sin(xy^2)$ .

3. Verify the theorem  $\frac{d^2u}{dx dy} = \frac{d^2u}{dy dx}$  when  $u = \log \tan(ax + y^2)$ .

4. Verify the theorem  $\frac{d^2 u}{dy^2 dx} = \frac{d^2 u}{dx dy^2}$  when  $u = \tan^{-1} \frac{x}{y}$ .

5. Verify the theorem  $\frac{d^2 u}{dy^2 dx^2} = \frac{d^2 u}{dx^2 dy}$  when  $u = y \log (1 + xy)$ .

6. Given  $u = \sin x \cos y$ , prove that

$$\frac{d^2 u}{dy^2 dx^2} = \frac{d^2 u}{dx^2 dy^2} = \frac{d^2 u}{dx dy dx dy}.$$

7. Given  $u = x^3 z^4 + \epsilon^2 y^3 z^3 + x^2 y^3 z^3$ , derive

$$\frac{d^2 u}{dx^2 dy dz} = 6y\epsilon^2 z^3 + 8yz.$$

8. Given  $u = \frac{1}{\sqrt{(4ab - c^2)}}$ , prove that

$$\frac{d^2 u}{dc^2} = \frac{d^2 u}{da db}.$$

9. Given  $u = (x + y)^2$ , prove that

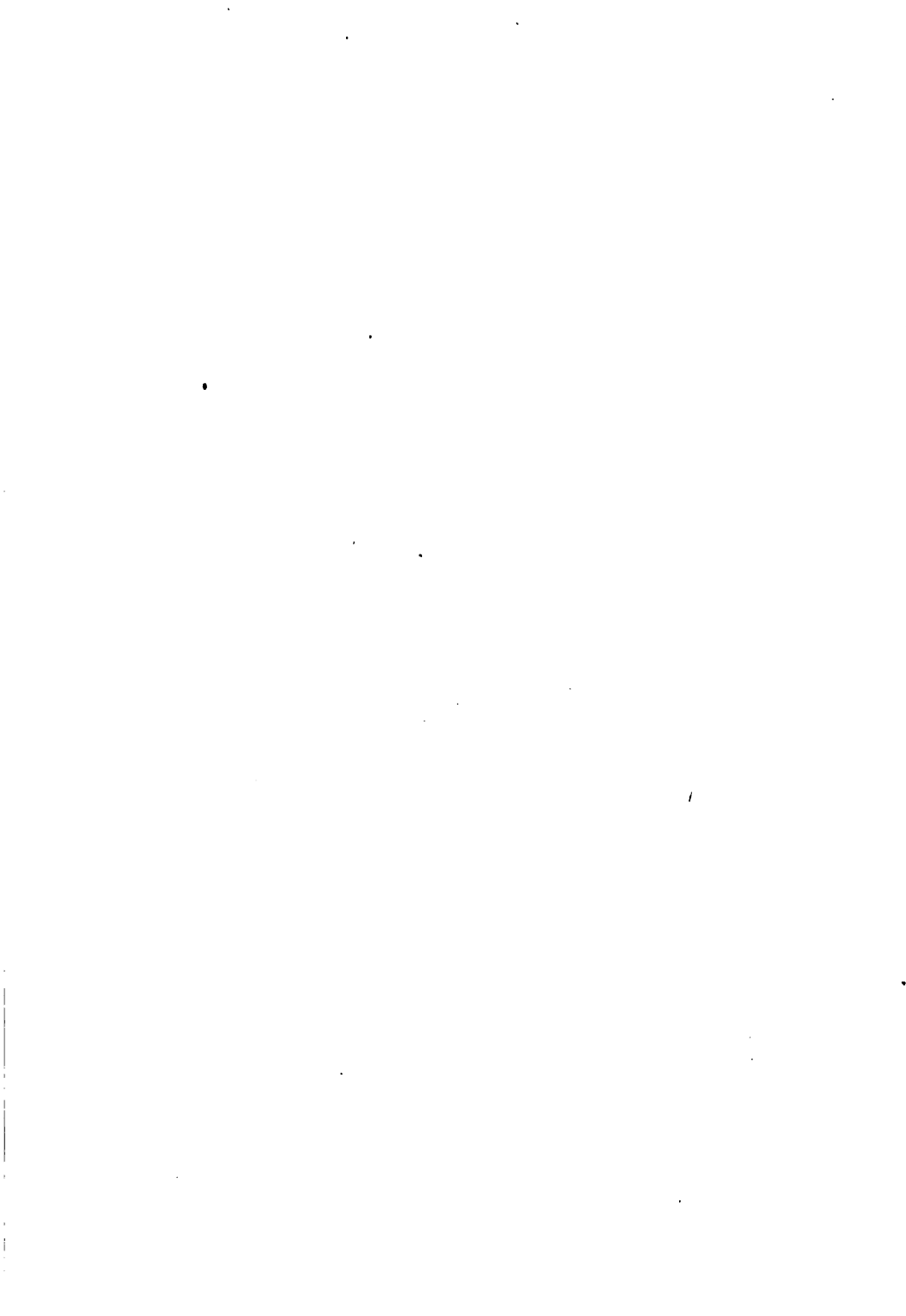
$$x \frac{d^2 u}{dx^2} + y \frac{d^2 u}{dx dy} = \frac{du}{dx}.$$

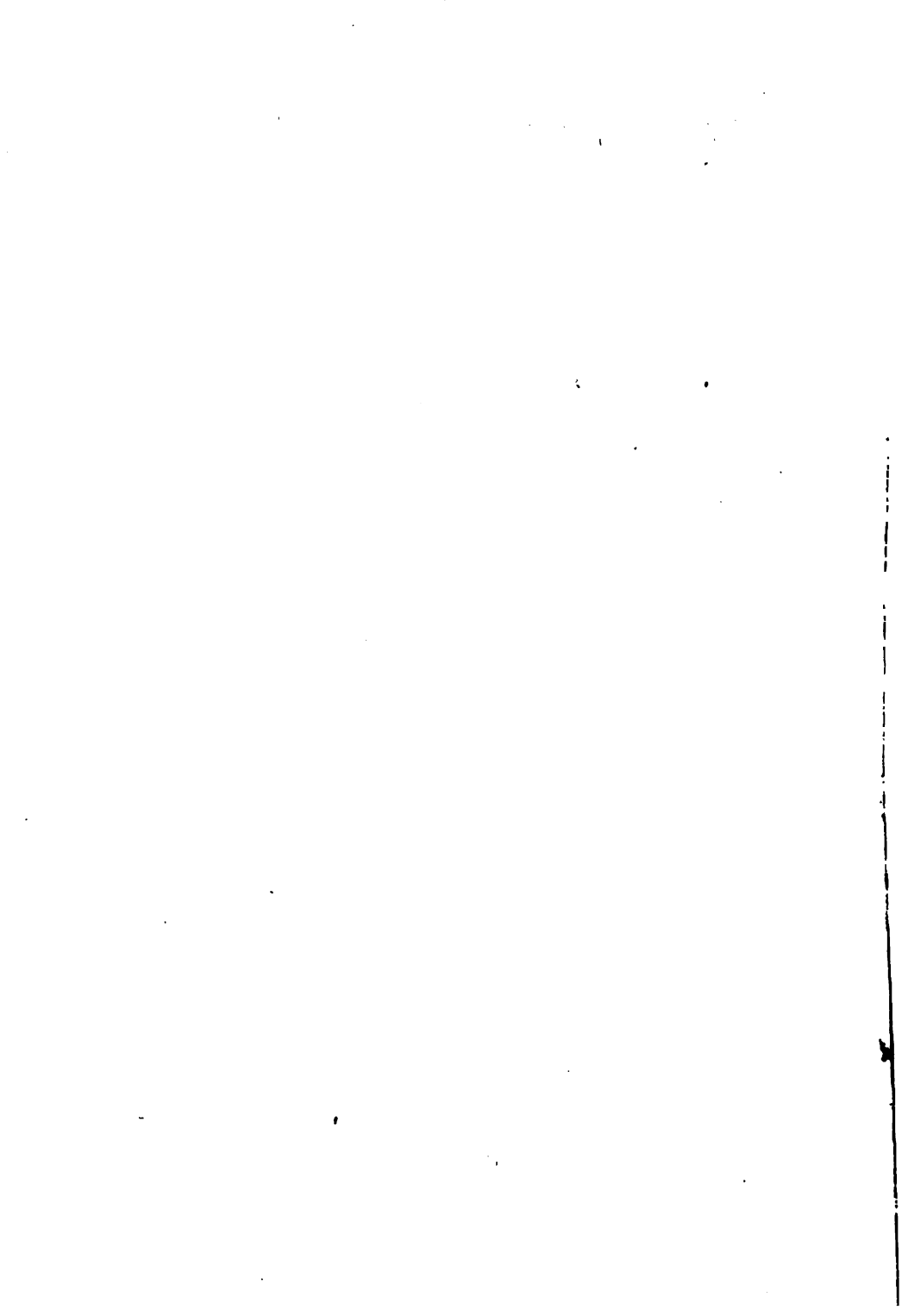
10. Given  $u = \frac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$ , prove that

$$\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} = 0.$$









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